2021.3 Question 12

1. Let X_i be the outcome of player i in a die roll. Then we have

$$X_{ij} = \begin{cases} 1, & X_i = X_j, \\ 0, & X_i \neq X_j. \end{cases}$$

Hence, we have

$$\begin{aligned} \mathrm{P}(X_{ij} = 1) &= \mathrm{P}(X_i = X_j) \\ &= \sum_{n=1}^{6} \mathrm{P}(X_i = X_j = n) \\ &= \sum_{n=1}^{6} \mathrm{P}(X_i = n)\, \mathrm{P}(X_j = n) \\ &= \sum_{n=1}^{6} \frac{1}{6} \cdot \frac{1}{6} \\ &= 6 \cdot \frac{1}{6} \cdot \frac{1}{6} \\ &= \frac{1}{6}, \end{aligned}$$

and hence $\mathrm{P}(X_{ij} = 0) = 1 - \frac{1}{6} = \frac{5}{6}$. Furthermore,

$$\mathrm{E}\left(X_{ij}\right) = \frac{1}{6} \cdot 1 = \frac{1}{6},$$

and hence

$$\mathrm{Var}\left(X_{ij}\right) = \mathrm{E}\left(X_{ij}^2\right) - \left(X_{ij}\right)^2 = \frac{1}{6} \cdot 1 - \left(\frac{1}{6}\right)^2 = \frac{5}{36}.$$

For any $1 \leq i < j < k \leq n$, we have

$$\begin{aligned} \mathrm{P}(X_{ij} = 1, X_{jk} = 1) &= \mathrm{P}(X_i = X_j, X_j = X_k) \\ &= \mathrm{P}(X_i = X_j = X_k) \\ &= \sum_{n=1}^{6} \mathrm{P}(X_i = X_j = X_k = n) \\ &= \sum_{n=1}^{6} \mathrm{P}(X_i = n)\, \mathrm{P}(X_j = n)\, \mathrm{P}(X_k = n) \\ &= \sum_{n=1}^{6} \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \\ &= 6 \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \\ &= \frac{1}{36} \\ &= \mathrm{P}(X_{ij} = 1)\, \mathrm{P}(X_{jk} = 1), \end{aligned}$$

$$\begin{aligned}
\mathrm{P}(X_{ij}=1,X_{jk}=0) &= \mathrm{P}(X_i=X_j,X_j\neq X_k)\\
&= \sum_{n=1}^{6}\sum_{m\neq n}\mathrm{P}(X_i=X_j=n,X_k=m)\\
&= \sum_{n=1}^{6}\sum_{m\neq n}\mathrm{P}(X_i=n)\,\mathrm{P}(X_j=n)\,\mathrm{P}(X_k=m)\\
&= \sum_{n=1}^{6}\sum_{m\neq n}\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\\
&= 6\cdot 5\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\\
&= \frac{5}{36}\\
&= \mathrm{P}(X_{ij}=1)\,\mathrm{P}(X_{jk}=0),
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}(X_{ij}=0,X_{jk}=1) &= \mathrm{P}(X_i\neq X_j,X_j=X_k)\\
&= \sum_{n=1}^{6}\sum_{m\neq n}\mathrm{P}(X_i=m,X_j=X_k=m)\\
&= \sum_{n=1}^{6}\sum_{m\neq n}\mathrm{P}(X_i=m)\,\mathrm{P}(X_j=n)\,\mathrm{P}(X_k=n)\\
&= \sum_{n=1}^{6}\sum_{m\neq n}\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\\
&= 6\cdot 5\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\\
&= \frac{5}{36}\\
&= \mathrm{P}(X_{ij}=0)\,\mathrm{P}(X_{jk}=1),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{P}(X_{ij}=0,X_{jk}=0) &= \mathrm{P}(X_i\neq X_j,X_j\neq X_k)\\
&= \sum_{n=1}^{6}\sum_{m\neq n}\sum_{l\neq n}\mathrm{P}(X_i=m,X_j=n,X_k=l)\\
&= \sum_{n=1}^{6}\sum_{m\neq n}\sum_{l\neq n}\mathrm{P}(X_i=m)\,\mathrm{P}(X_j=n)\,\mathrm{P}(X_k=l)\\
&= \sum_{n=1}^{6}\sum_{m\neq n}\sum_{l\neq n}\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\\
&= 6\cdot 5\cdot 5\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\\
&= \frac{25}{36}\\
&= \mathrm{P}(X_{ij}=0)\,\mathrm{P}(X_{jk}=0).
\end{aligned}$$

Hence, X_{ij} and X_{jk} are independent, and therefore X_{12} is independent of X_{23}.

Similarly, for $0\leq i<j<k\leq n$, we have X_{ij} is independent of X_{ik}, and X_{ik} is independent of X_{jk}. Furthermore, for $0\leq i<j\leq n$ and $0\leq k<p\leq n$, where none of i,j,k,l are equal, we have X_{ij} is independent of X_{kl} since the outcomes are completely irrelevant and independent.

Hence, X_{ij} s are pairwise independent. Let X be the total score:

$$X=\sum_{0\leq i<j\leq n}X_{ij}$$

and hence we have

$$\begin{aligned}
\mathrm{E}(X) &= \mathrm{E}\left(\sum_{0 \le i < j \le n} X_{ij}\right) \\
&= \sum_{0 \le i < j \le n} \mathrm{E}\left(X_{ij}\right) \\
&= \sum_{0 \le i < j \le n} \cdot \frac{1}{6} \\
&= \binom{n}{2} \cdot \frac{1}{6} \\
&= \frac{n(n-1)}{12},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{Var}\left(\sum_{0 \le i < j \le n} X_{ij}\right) \\
&= \sum_{0 \le i < j \le n} \mathrm{Var}\left(X_{ij}\right) \\
&= \sum_{0 \le i < j \le n} \cdot \frac{5}{36} \\
&= \binom{n}{2} \cdot \frac{5}{36} \\
&= \frac{5n(n-1)}{72},
\end{aligned}$$

2. Define

$$Y = \sum_{i=1}^{m} Y_i,$$

and hence

$$\mathrm{E}(Y) = \mathrm{E}\left(\sum_{i=1}^{m} Y_i\right) = \sum_{i=1}^{m} \mathrm{E}(Y_i) = 0.$$

Hence,

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{E}\left(Y^2\right) - \mathrm{E}(Y)^2 \\
&= \mathrm{E}\left(\left(\sum_{i=1}^{m} Y_i\right)^2\right) \\
&= \mathrm{E}\left(\sum_{i=1}^{m} Y_i^2 + \sum_{i \ne j} Y_i Y_j\right) \\
&= \mathrm{E}\left(\sum_{i=1}^{m} Y_i^2 + 2 \sum_{1 \le i < j \le m} Y_i Y_j\right) \\
&= \mathrm{E}\left(\sum_{i=1}^{m} Y_i^2 + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} Y_i Y_j\right) \\
&= \sum_{i=1}^{m} \mathrm{E}\left(Y_i^2\right) + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \mathrm{E}\left(Y_i Y_j\right),
\end{aligned}$$

as desired.

3. By definition, we have

$$Z_{ij} = \begin{cases} 1, & X_i = X_j \text{ is even,} \\ -1, & X_i = X_j \text{ is odd,} \\ 0, & X_i \neq X_j. \end{cases}$$

Hence, we have $\mathrm{P}(Z_{ij} = 0) = \mathrm{P}(X_{ij} = 0) = \frac{5}{6}$, and

$$\begin{aligned} \mathrm{P}(Z_{ij} = 1) = \mathrm{P}(Z_{ij} = -1) &= \frac{1}{2}\left(1 - \mathrm{P}(Z_{ij} = 0)\right) \\ &= \frac{1}{2}\left(1 - \mathrm{P}(X_{ij} = 0)\right) \\ &= \frac{1}{2}\left(1 - \frac{5}{6}\right) \\ &= \frac{1}{12}, \end{aligned}$$

which means $\mathrm{E}\left(Z_{ij}\right) = 0$.

Consider $Z_{12} = 1$ and $Z_{23} = -1$. If $Z_{12} = 1$ and $Z_{23} = -1$, this means $X_1 = X_2$ are both even, and $X_2 = X_3$ are both odd. This is impossible, and hence

$$\mathrm{P}(Z_{12} = 1, Z_{23} = -1) = 0.$$

On the other hand,

$$\mathrm{P}(Z_{12} = 1)\,\mathrm{P}(Z_{23} = -1) = \frac{1}{12} \cdot \frac{1}{12} = \frac{1}{144} \neq 0,$$

and so Z_{12} and Z_{23} are not independent.

Notice that $X_{ij} = Z_{ij}^2$ and so $\mathrm{E}\left(Z_{ij}^2\right) = \mathrm{E}(X_{ij}) = \frac{1}{6}$.

We can say for $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$, where none of i, j, k, l are equal, since X_i, X_j, X_k and X_l are independent, we must have Z_{ij} is independent of Z_{kl}, and hence

$$\mathrm{E}\left(Z_{ij} Z_{kl}\right) = \mathrm{E}\left(Z_{ij}\right)\mathrm{E}\left(Z_{kl}\right) = 0.$$

However, for $1 \leq i < j < k \leq n$, we have

$$\mathrm{P}(Z_{ij} Z_{jk} = -1) = \mathrm{P}(Z_{ij} = 1, Z_{jk} = -1) + \mathrm{P}(Z_{ij} = -1, Z_{jk} = 1) = 0.$$

For the event $Z_{ij} Z_{jk} = 1$, it must be $Z_{ij} = Z_{jk} = \pm 1$, which is the event $X_{ij} = X_{jk} = 1$, and hence

$$\mathrm{P}(Z_{ij} Z_{jk} = 1) = \mathrm{P}(X_{ij} = X_{jk} = 1) = \mathrm{P}(X_{ij} = 1)\,\mathrm{P}(X_{jk} = 1) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

Hence, the only remaining case is $Z_{ij} Z_{jk} = 0$ which gives

$$\mathrm{P}(Z_{ij} Z_{jk} = 0) = 1 - \frac{1}{36} = \frac{35}{36},$$

and hence

$$\mathrm{E}\left(Z_{ij} Z_{jk}\right) = \frac{1}{36}.$$

Let Z be the total score

$$Z = \sum_{1 \leq i < j \leq n} Z_{ij},$$

and hence

$$\mathrm{E}(Z) = \mathrm{E}\left(\sum_{1 \leq i < j \leq n} Z_{ij}\right) = \sum_{1 \leq i < j \leq n} \mathrm{E}\left(Z_{ij}\right) = 0.$$

For the variance, the second part of the sum consists of the non-repeating pairwise products of Z_{ij} and Z_{kl} for $1 \leq i, j, k, l \leq n$, $i < j$ and $k < l$, and finally for non-repeating, $i < k$ or $i = k$ and $j < l$. Let the indices be $1 \leq i < j < k \leq n$, and the pairs must be one of the following three

$$(Z_{ij}, Z_{ik}), (Z_{ij}, Z_{jk}), (Z_{ik}, Z_{jk})$$

and hence there are

$$3 \cdot \binom{n}{3} = \frac{n(n-1)(n-2)}{2}$$

such pairs.

Hence,

$$\begin{aligned}
\mathrm{Var}(Z) &= \sum_{1 \leq i < j \leq n} \mathrm{E}\left(Z_{ij}^2\right) + 2 \cdot \frac{n(n-1)(n-2)}{2} \cdot \frac{1}{36} \\
&= \binom{n}{2} \cdot \frac{1}{6} + \frac{n(n-1)(n-2)}{36} \\
&= \frac{n(n-1)}{12} + \frac{n(n-1)(n-2)}{36} \\
&= \frac{n(n-1)}{36} \cdot [3 + (n-2)] \\
&= \frac{n(n-1)}{36}(n+1) \\
&= \frac{n(n^2-1)}{36},
\end{aligned}$$

as desired.