2021.3 Question 11

1. From the definitions, $X \sim \mathrm{Exp}(\lambda)$, and $Y = \lfloor X \rfloor$.

   Hence, for $n \geq 0$,

$$\begin{aligned} \mathrm{P}(Y = n) &= \mathrm{P}(\lfloor X \rfloor = n) \\ &= \mathrm{P}(n \leq X < n+1) \\ &= \int_n^{n+1} f(x) \, \mathrm{d}x \\ &= \int_n^{n+1} \lambda \cdot e^{-\lambda x} \, \mathrm{d}x \\ &= \left[-e^{-\lambda x} \right]_n^{n+1} \\ &= -e^{-\lambda(n+1)} + e^{-\lambda n} \\ &= e^{-n\lambda} \left(1 - e^{-\lambda} \right), \end{aligned}$$

   as desired.

2. Since $Z = X - Y$, we know that $Z = \{X\}$ where $\{x\}$ stands for the fractional part of x.

   Hence, for $0 \leq z \leq 1$, we have

$$\begin{aligned} \mathrm{P}(Z < z) &= \mathrm{P}(\{X\} < z) \\ &= \mathrm{P}(X - Y < z) \\ &= \sum_{n=0}^{\infty} \mathrm{P}(X < Y + z, Y = n) \\ &= \sum_{n=0}^{\infty} \mathrm{P}(n \leq X < n + z) \\ &= \sum_{n=0}^{\infty} \int_n^{n+z} \lambda \cdot e^{-\lambda x} \, \mathrm{d}x \\ &= \sum_{n=0}^{\infty} \left[-e^{-\lambda x} \right]_n^{n+z} \\ &= \sum_{n=0}^{\infty} \left[-e^{-\lambda(n+z)} + e^{-\lambda n} \right] \\ &= \sum_{n=0}^{\infty} e^{-n\lambda} \left(1 - e^{-\lambda z} \right) \\ &= \left(1 - e^{-\lambda z} \right) \sum_{n=0}^{\infty} e^{-n\lambda} \\ &= \left(1 - e^{-\lambda z} \right) \cdot \frac{1}{1 - e^{-\lambda}} \\ &= \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}}, \end{aligned}$$

   as desired.

3. It must be the case that $0 \leq Z < 1$, and the cumulative distribution function of Z is given by, for $0 \leq z \leq 1$,

$$F_Z(z) = \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}}.$$

   By differentiating with respect to z, we get the probability density function of Z is given by, for

$0 \le z \le 1$,

$$\begin{aligned}f_Z(z) &= F_Z'(z) \\ &= \frac{\mathrm{d}}{\mathrm{d}z}\frac{1-e^{-\lambda z}}{1-e^{-\lambda}} \\ &= \frac{1}{1-e^{-\lambda}} \cdot \left(\lambda \cdot e^{-\lambda z}\right) \\ &= \frac{\lambda e^{-\lambda z}}{1-e^{-\lambda}},\end{aligned}$$

and zero everywhere else.

Hence, the expectation is given by

$$\begin{aligned}\mathrm{E}(Z) &= \int_0^1 z f_Z(z)\,\mathrm{d}z \\ &= \int_0^1 \frac{\lambda z e^{-\lambda z}}{1-e^{-\lambda}}\,\mathrm{d}z \\ &= \frac{\lambda}{1-e^{-\lambda}}\int_0^1 z e^{-\lambda z}\,\mathrm{d}z \\ &= -\frac{1}{1-e^{-\lambda}}\int_0^1 z\,\mathrm{d}e^{-\lambda z} \\ &= -\frac{1}{1-e^{-\lambda}}\left[\left(ze^{-\lambda z}\right)_0^1 - \int_0^1 e^{-\lambda z}\,\mathrm{d}z\right] \\ &= -\frac{1}{1-e^{-\lambda}}\left[ze^{-\lambda z} + \frac{e^{-\lambda z}}{\lambda}\right]_0^1 \\ &= -\frac{1}{1-e^{-\lambda}}\left[\left(e^{-\lambda} + \frac{e^{-\lambda}}{\lambda}\right) - \left(0 + \frac{1}{\lambda}\right)\right] \\ &= \frac{\frac{1}{\lambda} - \frac{e^{-\lambda}}{\lambda} - e^{-\lambda}}{1-e^{-\lambda}} \\ &= \frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{\lambda\left(1-e^{-\lambda}\right)}.\end{aligned}$$

4. Since $0 \le z_1 < z_2 \le 1$, we have $n \le n+z_1 < n+z_2 \le n+1$, and hence

$$\begin{aligned}\mathrm{P}(Y=n, z_1<Z<z_2) &= \mathrm{P}(Y=n, z_1 < X-Y < z_2) \\ &= \mathrm{P}(n+z_1 < X < n+z_2) \\ &= \int_{n+z_1}^{n+z_2} \lambda \cdot e^{-\lambda x} \\ &= \left[-e^{-\lambda x}\right]_{n+z_1}^{n+z_2} \\ &= e^{-\lambda(n+z_1)} - e^{-\lambda(n+z_2)} \\ &= e^{-\lambda n}\left[e^{-\lambda z_1} - e^{-\lambda z_2}\right].\end{aligned}$$

On the other hand, notice

$$\begin{aligned}\mathrm{P}(Y=n)\,\mathrm{P}(z_1<Z<z_2) &= \mathrm{P}(Y=n)\left(\mathrm{P}(Z<z_2) - \mathrm{P}(Z-z_1)\right) \\ &= (1-e^{-\lambda})e^{-n\lambda}\cdot\left[\frac{1-e^{-\lambda z_2}}{1-e^{-\lambda}} - \frac{1-e^{-\lambda z_1}}{1-e^{-\lambda}}\right] \\ &= e^{-n\lambda}\left[\left(1-e^{-\lambda z_2}\right) - \left(1-e^{-\lambda z_1}\right)\right] \\ &= e^{-n\lambda}\left[e^{-\lambda z_1} - e^{-\lambda z_2}\right].\end{aligned}$$

Hence, we have

$$\mathrm{P}(Y=n, z_1<Z<z_2) = \mathrm{P}(Y=n)\,\mathrm{P}(z_1<Z<z_2),$$

and we can conclude that Y and Z are independent.