2021.3 Question 1

1. By using the chain rule, we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t} \\ &= \frac{12\cos t - 12\sin^2 t\cos t}{12\cos^2 t\sin t} \\ &= \frac{\cos t - \sin^2 t\cos t}{\cos^2 t\sin t} \\ &= \frac{1-\sin^2 t}{\cos t\sin t} \\ &= \frac{\cos^2 t}{\cos t\sin t} \\ &= \frac{\cos t}{\sin t} \\ &= \cot t.\end{aligned}$$

Hence, at $t = \varphi$, the normal of this curve has gradient $-\tan\varphi$, and hence it has equation

$$\begin{aligned}y - \left(12\sin\varphi - 4\sin^3\varphi\right) &= -\tan\varphi\left(x - \left(-4\cos^3\varphi\right)\right) \\ y - 12\sin\varphi + 4\sin^3\varphi &= -\tan\varphi x - 4\cos^3\varphi\tan\varphi \\ \cos\varphi y - 12\sin\varphi\cos\varphi + 4\sin^3\varphi\cos\varphi &= -\sin\varphi x - 4\cos^3\varphi\sin\varphi \\ \sin\varphi x + \cos\varphi y &= 12\sin\varphi\cos\varphi - 4\sin^3\varphi\cos\varphi - 4\cos^3\varphi\sin\varphi \\ \sin\varphi x + \cos\varphi y &= 4\sin\varphi\cos\varphi\left(3 - \sin^2\varphi - \cos^2\varphi\right) \\ \sin\varphi x + \cos\varphi y &= 8\sin\varphi\cos\varphi.\end{aligned}$$

The curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ can be parametrised as $x = 8\cos^3 t$ and $y = 8\sin^3 t$:

$$\begin{aligned}x^{\frac{2}{3}} + y^{\frac{2}{3}} &= \left(8\cos^3 t\right)^{\frac{2}{3}} + \left(8\sin^3 t\right)^{\frac{2}{3}} \\ &= 4\cos^2 t + 4\sin^2 t \\ &= 4.\end{aligned}$$

Hence, the gradient of the tangent at a point is

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t} \\ &= \frac{24\sin^2 t\cos t}{-24\cos^2 t\sin t} \\ &= -\tan t,\end{aligned}$$

and the equation of the tangent at the point $t = \varphi$ is

$$\begin{aligned}y - 8\sin^3\varphi &= -\tan\varphi\left(x - 8\cos^3\varphi\right) \\ \cos\varphi y - 8\sin^3\varphi\cos\varphi &= -\sin\varphi x + 8\cos^3\varphi\sin\varphi \\ \sin\varphi x + \cos\varphi y &= 8\sin\varphi\cos\varphi\left(\sin^2\varphi + \cos^2\varphi\right) \\ \sin\varphi x + \cos\varphi y &= 8\sin\varphi\cos\varphi,\end{aligned}$$

which shows the normal to the original curve is the tangent to this new curve at $(8\cos^3\varphi, 8\sin^3\varphi)$.

2. By using the chain rule, we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t} \\ &= \frac{\cos t - \cos t + t\sin t}{-\sin t + \sin t + t\cos t} \\ &= \frac{t\sin t}{t\cos t} \\ &= \tan t.\end{aligned}$$

Hence, at $t = \varphi$, the normal of this curve has gradient $-\cot\varphi$, and hence it has equation

$$\begin{aligned}
y - (\sin\varphi - \varphi\cos\varphi) &= -\cot\varphi\,(x - (\cos\varphi + \varphi\sin\varphi)) \\
\sin\varphi y - \sin^2\varphi + \varphi\sin\varphi\cos\varphi &= -\cos\varphi x + \cos^2\varphi + \varphi\sin\varphi\cos\varphi \\
\cos\varphi x + \sin\varphi y &= \sin^2\varphi + \cos^2\varphi \\
\cos\varphi x + \sin\varphi y &= 1.
\end{aligned}$$

The distance of this normal to the origin is

$$\frac{|\cos\varphi\cdot 0 + \sin\varphi\cdot 0 - 1|}{\sqrt{\cos^2\varphi + \sin^2\varphi}} = 1,$$

which is a constant, and hence this curve is tangent to the unit circle $x^2 + y^2 = 1$.