2020.3 Question 7

1. By differentiating both sides of the second differential equation, we can see

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+g'(x)y+g(x)\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{\mathrm{d}u}{\mathrm{d}x}\\&=h(x)-f(x)u\\&=h(x)-f(x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}+g(x)y\right),\end{aligned}$$

and hence rearranging gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(f(x)+g(x))\frac{\mathrm{d}y}{\mathrm{d}x}+(g'(x)+f(x)g(x))y=f(x),$$

as desired.

2. We must have $g(x)+f(x)=1+\frac{4}{x}$, and $g'(x)+f(x)g(x)=\frac{2}{x}+\frac{2}{x^2}$, with $h(x)=4x+12$.

   Hence, from the first equation, we have $f(x)=1+\frac{4}{x}-g(x)$, and putting this into the second equation gives us

$$g'(x)+\left(1+\frac{4}{x}-g(x)\right)g(x)=\frac{2}{x}+\frac{2}{x^2}.$$

   If $g(x)=kx^n$, then $g'(x)=knx^{n-1}$, and putting this back we have

$$knx^{n-1}+\left(1+\frac{4}{x}-kx^n\right)kx^n=\frac{2}{x}+\frac{2}{x^2},$$

   which gives

$$-k^2x^{2n}+kx^n+k(n+4)x^{n-1}=2x^{-1}+2x^{-2}.$$

   Therefore, we could simply let $n=-1$, and $k=2$. Verify that

$$\text{LHS}=-4x^{-2}+2x^{-1}+2\cdot 3x^{-2}=2x^{-1}+2x^{-2}=\text{RHS}.$$

   Hence, $g(x)=\frac{2}{x}$, and $f(x)=1+\frac{2}{x}$.

   The differential equation for u is

$$\frac{\mathrm{d}u}{\mathrm{d}x}+\left(1+\frac{2}{x}\right)u=4x+12.$$

   The integration factor is

$$I(x)=e^{\int(1+\frac{2}{x})\,\mathrm{d}x}=e^{x+2\ln x}=x^2e^x,$$

   and hence

$$x^2e^x\frac{\mathrm{d}u}{\mathrm{d}x}+e^x(x^2+2x)u=\frac{\mathrm{d}x^2e^xu}{\mathrm{d}x}=4x^3e^x+12x^2e^x.$$

   Notice the right-hand side is the derivative of $4x^3e^x$, and hence

$$x^2e^xu=4x^3e^x+C.$$

   When $x=1$,

$$\begin{aligned}u|_{x=1}&=\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1}+g(1)\;y|_{x=1}\\&=-3+2\cdot 5\\&=7,\end{aligned}$$

   and hence

$$7e=4e+C,$$

giving $C = 3e$.

Hence,

$$u = 4x + \frac{3e}{x^2 e^x}.$$

The differential equation for y gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = u,$$

and hence the integration factor is x^2, giving

$$\frac{\mathrm{d}x^2 y}{\mathrm{d}x} = 4x^3 + 3e \cdot e^{-x},$$

and hence by integration on both sides, we have

$$x^2 y = x^4 - 3e \cdot e^{-x} + C'.$$

Since when $x = 1$, $y = 5$, we must have

$$5 = 1 - 3 + C',$$

giving $C' = 7$.

Hence,

$$x^2 y = x^4 - 3e^{1-x} + 7,$$

and hence

$$y = x^2 - 3x^{-2}e^{1-x} + 7x^{-2}.$$