2020.3 Question 6

1. Note that this function has symmetry about the y-axis since cos is an even function.

   When $x = 0$, $y = 1 + \sqrt{1} = 2$. When $x = \pm\frac{\pi}{4}$, $y = \frac{1}{\sqrt{2}}$.

   We investigate the gradient:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin x - 2\sin 2x \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\cos 2x}} \\ &= -\sin x - \frac{\sin 2x}{\sqrt{\cos 2x}},\end{aligned}$$

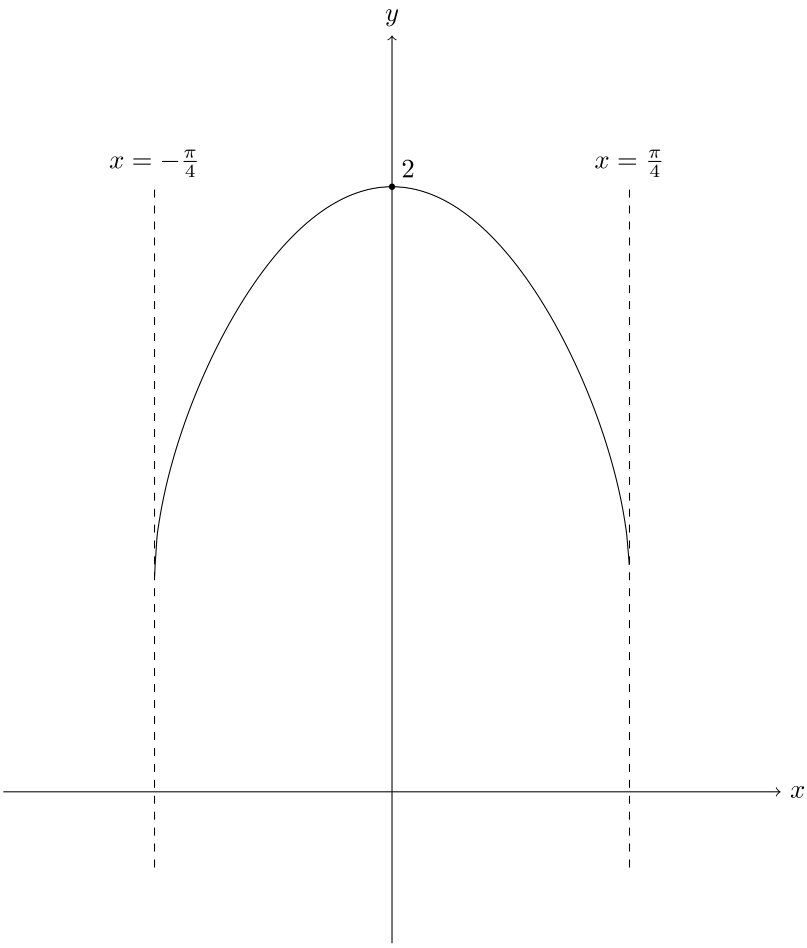
   so $\frac{\mathrm{d}y}{\mathrm{d}x}$ takes opposite sign as x, which means that y is decreasing when $x > 0$, and y is increasing when $x < 0$, and $x = 0$ gives a maximum.

   Also, note that

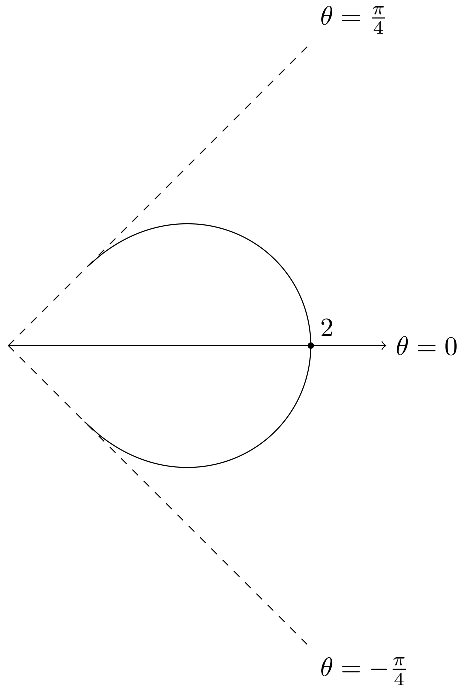
$$\lim_{x \to \frac{\pi}{4}^-} \frac{\mathrm{d}y}{\mathrm{d}x} = -\infty, \lim_{x \to -\frac{\pi}{4}^+} \frac{\mathrm{d}y}{\mathrm{d}x} = \infty,$$

   which means the tangent to the graph at those points are vertical.

   Hence, the graph looks as follows:



2. The graph looks as follows.



3. By solving the quadratic, we have

$$r = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4\sin^2\theta}}{2} = \cos\theta \pm \sqrt{\cos 2\theta}.$$

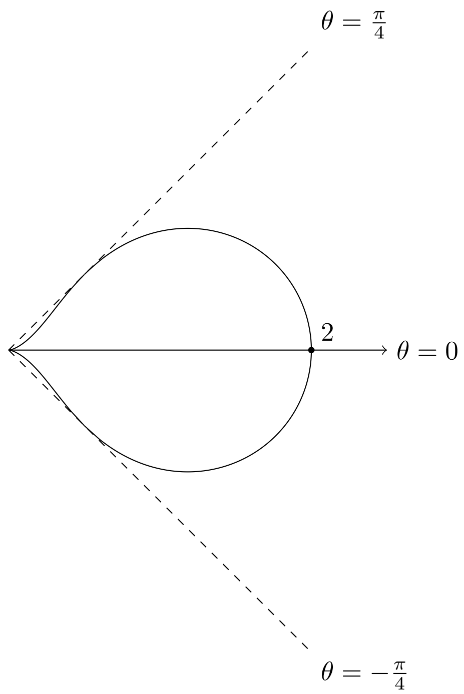
Hence, at $\theta = \pm\frac{1}{4}\pi, r = \frac{1}{\sqrt{2}}$.

When r is small, we must have that $r = \cos\theta - \sqrt{\cos 2\theta}$ and θ is small, and

$$\begin{aligned} -2r\cos\theta + \sin^2\theta &\approx 0 \\ r &\approx \frac{\sin^2\theta}{2\cos\theta} \\ r &\approx \frac{1}{2}\sin\theta\tan\theta \\ r &\approx \frac{1}{2}\theta^2, \end{aligned}$$

as desired.

The curve will look as follows. At $\theta = \pm\frac{1}{4}\pi$, the curve is tangent to the lines. At $r = 0$, the curves are tangent to the initial line.



The area between C_2 and $\theta = 0$ above the line is given by

$$\begin{aligned} A &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left[\left(\cos\theta + \sqrt{\cos 2\theta} \right)^2 - \left(\cos\theta - \sqrt{\cos 2\theta} \right)^2 \right] \mathrm{d}\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{4}} 4 \cos\theta \sqrt{\cos 2\theta} \, \mathrm{d}\theta \\ &= 2 \int_0^{\frac{\pi}{4}} \cos\theta \sqrt{\cos 2\theta} \, \mathrm{d}\theta \\ &= 2 \int_0^{\frac{\pi}{4}} \cos\theta \sqrt{1 - 2\sin^2\theta} \, \mathrm{d}\theta \\ &= 2 \int_0^{\frac{\pi}{4}} \sqrt{1 - 2\sin^2\theta} \, \mathrm{d}\sin\theta \\ &= 2 \int_0^{\frac{1}{\sqrt{2}}} \sqrt{1 - 2x^2} \, \mathrm{d}x \\ &= \sqrt{2} \int_0^1 \sqrt{1 - y^2} \, \mathrm{d}y \\ &= \sqrt{2} \cdot \frac{\pi}{4} \\ &= \frac{\pi}{2\sqrt{2}}, \end{aligned}$$

as desired, the final integral being because this is $\frac{1}{4}$ of the area of the unit circle, which is π.