2020.3 Question 5

We notice that

$$\begin{aligned}
\text{RHS} &= (x-y)\sum_{r=1}^{n} x^{n-r}y^{r-1} \\
&= x\sum_{r=1}^{n} x^{n-r}y^{r-1} - y\sum_{r=1}^{n} x^{n-r}y^{r-1} \\
&= \sum_{r=1}^{n} x^{n-r+1}y^{r-1} - \sum_{r=1}^{n} x^{n-r}y^{r} \\
&= \sum_{r=0}^{n-1} x^{n-r}y^{r} - \sum_{r=1}^{n} x^{n-r}y^{r} \\
&= x^n y^0 + \sum_{r=1}^{n-1} x^{n-r}y^{r} - \sum_{r=1}^{n-1} x^{n-r}y^{r} - x^0 y^n \\
&= x^n - y^n.
\end{aligned}$$

1. Notice that

$$\begin{aligned}
f(x) &= x^n \cdot \left(F(x) - \frac{A}{x-k}\right) \\
&= x^n \cdot \left(\frac{1}{x^n(x-k)} - \frac{A}{x-k}\right) \\
&= \frac{1}{x-k} - \frac{Ax^n}{x-k} \\
&= \frac{1-Ax^n}{x-k}.
\end{aligned}$$

Since f is a polynomial, the numerator must be divisible by the denominator, and hence when $x = k$, the numerator must be 0, which means

$$1 - Ak^n = 0,$$

and hence

$$A = \frac{1}{k^n}.$$

Hence,

$$f(x) = \frac{1-Ax^n}{x-k} = \frac{1}{x-k}\left(1 - \left(\frac{x}{k}\right)^n\right),$$

as desired.

Using the identity, we have

$$\begin{aligned}
f(x) &= \frac{1-Ax^n}{x-k} \\
&= \frac{1^n - \left(\frac{x}{k}\right)^n}{x-k} \\
&= \frac{1}{k^n}\cdot\frac{k^n - x^n}{x-k} \\
&= \frac{1}{k^n}\cdot\frac{-(x-k)\sum_{r=1}^{n} k^{n-r}x^{r-1}}{x-k} \\
&= -\sum_{r=1}^{n} k^{-r}x^{r-1},
\end{aligned}$$

and hence

$$\begin{aligned}F(x) &= \frac{A}{x-k} + \frac{f(x)}{x^n} \\ &= \frac{1}{k^n(x-k)} - \frac{\sum_{r=1}^n k^{-r}x^{r-1}}{x^n} \\ &= \frac{1}{k^n(x-k)} - \sum_{r=1}^n \frac{1}{k^r x^{n-r+1}} \\ &= \frac{1}{k^n(x-k)} - \sum_{r=1}^n \frac{1}{k^{n-r+1}x^r} \\ &= \frac{1}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^n \frac{1}{k^{n-r}x^r},\end{aligned}$$

as desired.

2. Notice that on one hand,

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n F(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x-k} = -\frac{1}{(x-k)^2},$$

and on the other hand, using the expression above, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}x^n F(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^n}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^n \frac{x^{n-r}}{k^{n-r}}\right] \\ &= \frac{nx^{n-1}k^n(x-k) - x^n k^n}{k^{2n}(x-k)^2} - \frac{1}{k}\sum_{r=1}^n \frac{(n-r)x^{n-r-1}}{k^{n-r}} \\ &= \frac{nx^{n-1}(x-k) - x^n}{k^n(x-k)^2} - \sum_{r=1}^n \frac{n-r}{k^{n-r+1}x^{-n+r+1}} \\ &= \frac{nx^{n-1}}{k^n(x-k)} - \frac{x^n}{k^n(x-k)^2} - \sum_{r=1}^n \frac{n-r}{k^{n-r+1}x^{-n+r+1}}.\end{aligned}$$

Hence,

$$\begin{aligned}-\frac{1}{(x-k)^2} &= \frac{nx^{n-1}}{k^n(x-k)} - \frac{x^n}{k^n(x-k)^2} - \sum_{r=1}^n \frac{n-r}{k^{n-r+1}x^{-n+r+1}} \\ \frac{1}{(x-k)^2} &= \frac{x^n}{k^n(x-k)^2} - \frac{nx^{n-1}}{k^n(x-k)} + \sum_{r=1}^n \frac{n-r}{k^{n-r+1}x^{-n+r+1}} \\ \frac{1}{x^n(x-k)^2} &= \frac{1}{k^n(x-k)^2} - \frac{n}{k^n x(x-k)} + \sum_{r=1}^n \frac{n-r}{k^{n-r+1}x^{r+1}},\end{aligned}$$

precisely as desired.

3. Let $n = 3$ and $k = 1$, and hence we have

$$\begin{aligned}\frac{1}{x^3(x-1)^2} &= \frac{1}{(x-1)^2} - \frac{3}{x(x-1)} + \sum_{r=1}^3 \frac{3-r}{x^{r+1}} \\ &= \frac{1}{(x-1)^2} - \frac{3}{x-1} + \frac{3}{x} + \frac{2}{x^2} + \frac{1}{x^3}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\int_2^N \frac{\mathrm{d}x}{x^3(x-1)^2} &= \int_2^N \left[\frac{1}{(x-1)^2} - \frac{3}{x-1} + \frac{3}{x} + \frac{2}{x^2} + \frac{1}{x^3}\right] \mathrm{d}x \\ &= \left[-\frac{1}{x-1} - 3\ln|x-1| + 3\ln|x| - \frac{2}{x} - \frac{1}{2x^2}\right]_2^N \\ &= \left(3\ln\frac{N}{N-1} - \frac{1}{N-1} - \frac{2}{N} - \frac{1}{2N^2}\right) - \left(3\ln\frac{2}{1} - \frac{1}{1} - \frac{2}{2} - \frac{1}{2\cdot 4}\right) \\ &= \frac{17}{8} - 3\ln 2 + \left(3\ln\frac{N}{N-1} - \frac{1}{N-1} - \frac{2}{N} - \frac{1}{2N^2}\right).\end{aligned}$$

We take the limit as $N \to \infty$. Since $\frac{N}{N-1} = 1 + \frac{1}{N-1}$, and as $N \to \infty$, $\frac{1}{N-1} \to 0$, this means that $\ln\frac{N}{N-1} \to 0$. All the fractions with N on the denominator also approaches 0. Hence, the limit of this integral as $N \to \infty$ is

$$\int_2^\infty \frac{\mathrm{d}x}{x^3(x-1)^2} = \frac{17}{8} - 3\ln 2.$$