2020.3 Question 3

1. Let k represent the point K in the complex plane, we must have

$$k - a = (b - a)\exp\left(-i\frac{\pi}{3}\right),$$

and hence

$$k = a + (b - a)\exp\left(-i\frac{\pi}{3}\right).$$

Notice that

$$\omega = \exp\left(\frac{i\pi}{6}\right) = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2},$$

and hence

$$\omega^* = \frac{\sqrt{3}}{2} - i\frac{1}{2}$$

Hence, g_{ab} is given by

$$\begin{aligned} g_{ab} &= \frac{a+b+k}{3} \\ &= \frac{a+b+a+(b-a)\exp\left(-i\frac{\pi}{3}\right)}{3} \\ &= \frac{2a+b+(b-a)\left[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right]}{3} \\ &= \frac{2a+b+(b-a)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)}{3} \\ &= \frac{\left(\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)a + \left(\frac{3}{2} - i\frac{\sqrt{3}}{2}\right)b}{3} \\ &= \frac{1}{\sqrt{3}} \cdot \left[\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)a + \left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)b\right] \\ &= \frac{1}{\sqrt{3}} \cdot (\omega a + \omega^* b), \end{aligned}$$

as desired.

2. Q_2 is a parallelogram if and only if

$$\begin{aligned} g_{bc} - g_{ab} &= g_{cd} - g_{da} \\ \frac{1}{\sqrt{3}}(\omega b + \omega^* c) - \frac{1}{\sqrt{3}}(\omega a + \omega^* b) &= \frac{1}{\sqrt{3}}(\omega c + \omega^* d) - \frac{1}{\sqrt{3}}(\omega d + \omega^* a) \\ \omega(b - a - c + d) &= \omega^*(d - a - c + b) \\ (\omega - \omega^*)\left[(b-a) - (c-d)\right] &= 0 \\ (b-a) - (c-d) &= 0 \\ b - a &= c - d, \end{aligned}$$

which is true if and only if Q_1 is a parallelogram. All the steps above are reversible. In particular, $\omega - \omega^* \neq 0$ so we can divide by $\omega - \omega^*$ on both sides.

3. Notice that

$$\begin{aligned} g_{bc} - g_{ab} &= \frac{1}{\sqrt{3}}(\omega b + \omega^* c) - \frac{1}{\sqrt{3}}(\omega a + \omega^* b) \\ &= \frac{1}{\sqrt{3}}\left[\omega^* c - \omega a + (\omega - \omega^*)b\right] \\ &= \frac{1}{\sqrt{3}}\left[\omega^* c - \omega a + bi\right], \end{aligned}$$

and that

$$\begin{aligned}
g_{ca} - g_{ab} &= \frac{1}{\sqrt{3}}\left(\omega c + \omega^* a\right) - \frac{1}{\sqrt{3}}\left(\omega a + \omega^* b\right) \\
&= \frac{1}{\sqrt{3}}\left[\omega c + (\omega^* - \omega)a - \omega^* b\right] \\
&= \frac{1}{\sqrt{3}}\left[\omega c - ai - \omega^* b\right].
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{\omega^*}{\omega} &= \frac{\exp\left(-\frac{i\pi}{6}\right)}{\exp\left(\frac{i\pi}{6}\right)} = \exp\left(-\frac{i\pi}{3}\right), \\
\frac{-\omega}{-i} &= \frac{\omega}{i} = \frac{\exp\left(\frac{i\pi}{6}\right)}{\exp\left(\frac{i\pi}{2}\right)} = \exp\left(-\frac{i\pi}{3}\right), \\
\frac{i}{-\omega^*} &= \frac{-i}{\omega^*} = \frac{\exp\left(-\frac{i\pi}{2}\right)}{\exp\left(-\frac{i\pi}{6}\right)} = \exp\left(-\frac{i\pi}{3}\right),
\end{aligned}$$

and hence we can wee

$$g_{bc} - g_{ab} = (g_{ca} - g_{ab}) \exp\left(-\frac{i\pi}{3}\right),$$

which means G_{BC} is the image of G_{CA} under rotation through $\frac{\pi}{3}$ clockwise about G_{AB}, and this shows that T_2 is an equilateral triangle.