2020.3 Question 2

1. We differentiate with respect to x on both sides, and we have

$$\cosh x + \cosh y \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

If the curve has a stationary point (x, y), we must have $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at that point, and hence

$$\cosh x = 0.$$

This is impossible since the cosh function has a range of $[1, +\infty)$, and hence C has no stationary points. (In fact we must have $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.)

Differentiating this with respect to x again gives

$$\sinh x + \sinh y \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \cosh y \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0.$$

At point (x, y), $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$ if and only if

$$\sinh x + \sinh y \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0.$$

From the previous differentiation, we know that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\cosh x}{\cosh y},$$

and hence

$$\sinh x + \sinh y \cdot \frac{\cosh^2 x}{\cosh^2 y} = 0,$$

which gives

$$\cosh^2 y \sinh x + \sinh y \cosh^2 x = 0.$$

Using the identity $\cosh^2 t = 1 + \sinh^2 t$, we have

$$\sinh x + \sinh^2 y \sinh x + \sinh y + \sinh^2 x \sinh y = 0,$$

and hence

$$(\sinh x + \sinh y)(1 + \sinh x \sinh y) = 0.$$

Since $\sinh x + \sinh y = 2k$ and k is positive, we can conclude that

$$1 + \sinh x \sinh y = 0,$$

as desired.

The only-if direction is identical since all steps above are reversible.

For a point of inflection, we must first have $\frac{\mathrm{d}^y x}{\mathrm{d}=^y}0$, and hence

$$\sinh x \sinh y = -1, \sinh x + \sinh y = 2k.$$

This means that $\sinh x$ and $\sinh y$ are roots to the quadratic equation in t

$$t^2 - 2kt - 1 = 0.$$

This equation solves to

$$t_{1,2} = \frac{2k \pm \sqrt{4k^2 + 4}}{2} = k \pm \sqrt{k^2 + 1}.$$

Therefore, the points where the second derivative is zero on the curve are

$$\left(\operatorname{arsinh}\left(k \pm \sqrt{k^2 + 1}\right), \operatorname{arsinh}\left(k \mp \sqrt{k^2 + 1}\right)\right).$$

2. If $x + y = a$ and $\sinh x + \sinh y = 2k$, we must have $y = a - x$, and hence

$$\begin{aligned}\frac{e^x - e^{-x}}{2} + \frac{e^{a-x} - e^{x-a}}{2} &= 2k \\ e^{2x} - 1 + e^a - e^{2x-a} &= 4ke^x \\ e^{2x}(1 - e^{-a}) - 4ke^x + (e^a - 1) &= 0,\end{aligned}$$

as desired.

Since e^x is always real, we must have

$$\begin{aligned}(-4k)^2 - 4(1 - e^{-a})(e^a - 1) &= 16k^2 - 4(e^a - 1 - 1 + e^{-a}) \\ &= 16k^2 - 4(2\cosh a - 2) \\ &= 16k^2 + 8 - 8\cosh a \\ &\geq 0,\end{aligned}$$

and hence

$$\cosh a \leq 2k^2 + 1.$$

As for the left-hand side inequality, we already know $\cosh a \geq 1$. $\cosh a = 1$ if and only if $a = x + y = 0$, in which case

$$\sinh x + \sinh y = \sinh x + \sinh(-x) = 0 \neq 2k,$$

since $2k > 0$.

Hence, we must have

$$1 < \cosh a \leq 2k^2 + 1,$$

as desired.

3. Notice that when $\cosh a = 2k^2 + 1$, there is precisely one root to the quadratic equation, which means $x = y$. Hence,

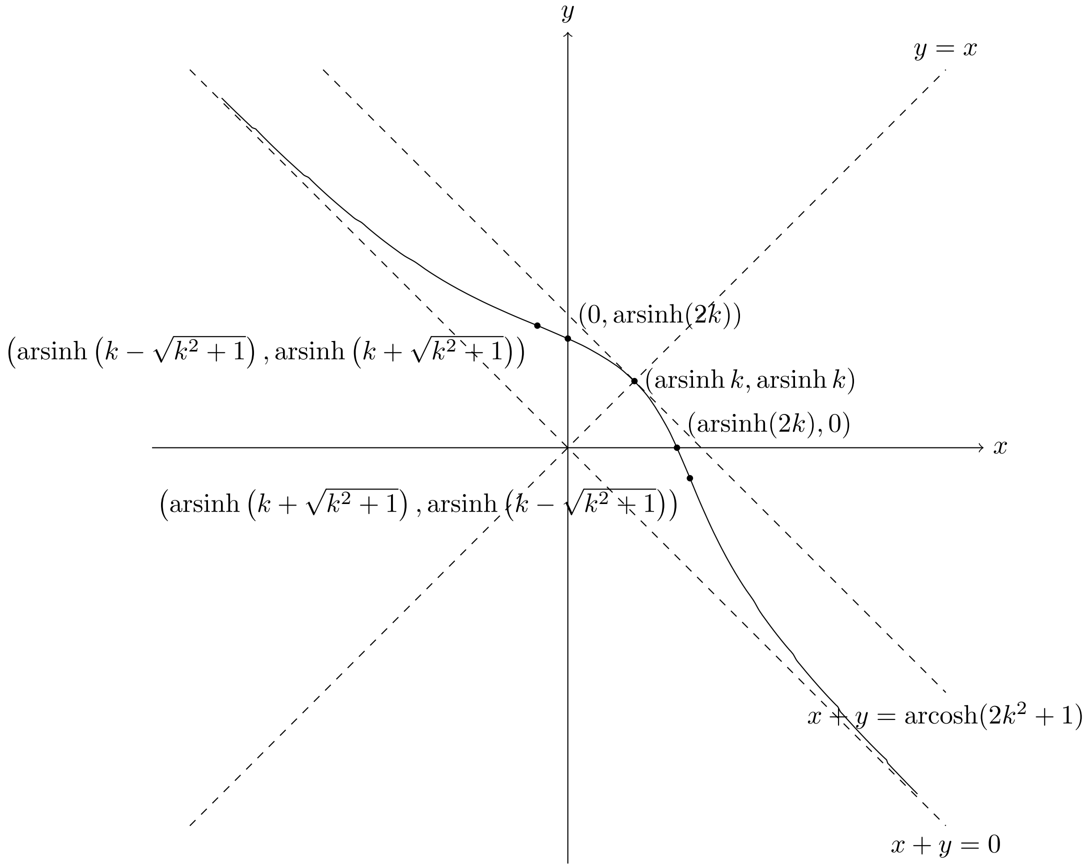
$$\begin{aligned}2k^2 + 1 &= \cosh a \\ &= \cosh(x + y) \\ &= \cosh x \cosh y + \sinh x \sinh y \\ &= \cosh^2 x + \sinh^2 x \\ &= 1 + 2\sinh^2 x,\end{aligned}$$

which shows that (since $\sinh x + \sinh y = k$)

$$\sinh x = \sinh y = k.$$

The graph meets the axis at $(0, \operatorname{arsinh}(2k))$ and $(\operatorname{arsinh}(2k), 0)$.

Hence, the graph must look as follows:



y
$y = x$
$(0, \operatorname{arsinh}(2k))$
$\left(\operatorname{arsinh}\left(k - \sqrt{k^2+1}\right), \operatorname{arsinh}\left(k + \sqrt{k^2+1}\right)\right)$
$(\operatorname{arsinh} k, \operatorname{arsinh} k)$
$(\operatorname{arsinh}(2k), 0)$
x
$\left(\operatorname{arsinh}\left(k + \sqrt{k^2+1}\right), \operatorname{arsinh}\left(k - \sqrt{k^2+1}\right)\right)$
$x + y = \operatorname{arcosh}(2k^2+1)$
$x + y = 0$