2020.3 Question 12

1. By the definition within the question, we have that $X, Y \sim \text{Geo}(p)$, and for $t \geq 1$,

$$\mathrm{P}(X = t) = \mathrm{P}(Y = t) = q^{t-1}p.$$

For $S = X + Y$, we have for $s \geq 2$,

$$\begin{aligned}
\mathrm{P}(S = s) &= \mathrm{P}(X + Y = s) \\
&= \sum_{t=1}^{s-1} \mathrm{P}(X = t, Y = s - t) \\
&= \sum_{t=1}^{s-1} \mathrm{P}(X = t)\,\mathrm{P}(Y = s - t) \\
&= \sum_{t=1}^{s-1} q^{t-1}pq^{s-t-1}p \\
&= \sum_{t=1}^{s-1} q^{s-2}p^2 \\
&= (s-1)q^{s-2}p^2.
\end{aligned}$$

For $T = \max\{X, Y\}$, we have for $t \geq 1$,

$$\begin{aligned}
\mathrm{P}(T = t) &= \mathrm{P}(X = Y = t) + \mathrm{P}(X = t, Y < X) + \mathrm{P}(Y = t, X < Y) \\
&= \mathrm{P}(X = t, Y = t) + 2\,\mathrm{P}(X = t, Y < X) \\
&= \mathrm{P}(X = t)\,\mathrm{P}(Y = t) + 2\sum_{r=1}^{t-1} \mathrm{P}(X = t, Y = r) \\
&= \mathrm{P}(X = t)\,\mathrm{P}(Y = t) + 2\sum_{r=1}^{t-1} \mathrm{P}(X = t)\,\mathrm{P}(Y = r) \\
&= q^{t-1}pq^{t-1}p + 2q^{t-1}p\sum_{r=1}^{t-1} q^{r-1}p \\
&= q^{2t-2}p^2 + 2q^{t-1}p^2\sum_{r=1}^{t-1} q^{r-1} \\
&= q^{2t-2}p^2 + 2q^{t-1}p^2\frac{1 - q^{t-1}}{1 - q} \\
&= q^{2t-2}p^2 + 2q^{t-1}p^2\frac{1 - q^{t-1}}{p} \\
&= q^{2t-2}p^2 + 2q^{t-1}p(1 - q^{t-1}) \\
&= pq^{t-1}\left(pq^{t-1} + 2 - 2q^{t-1}\right) \\
&= pq^{t-1}\left((1-q)q^{t-1} + 2 - 2q^{t-1}\right) \\
&= pq^{t-1}\left(2 + q^t - q^{t-1}\right)
\end{aligned}$$

2. Since $U = |X - Y|$, we have $U \geq 0$. For $u \geq 1$, we have

$$\begin{aligned}
\mathrm{P}(U = u) &= \mathrm{P}(|X - Y| = u) \\
&= \mathrm{P}(X - Y = \pm u) \\
&= 2\,\mathrm{P}(X - Y = u) \\
&= 2\sum_{t=1}^{\infty} \mathrm{P}(X = u + t, Y = t) \\
&= 2\sum_{t=1}^{\infty} \mathrm{P}(X = u + t)\,\mathrm{P}(Y = t) \\
&= 2\sum_{t=1}^{\infty} q^{u+t-1} p q^{t-1} p \\
&= 2q^u p^2 \sum_{t=1}^{\infty} q^{2t-2} \\
&= 2q^u p^2 \cdot \frac{1}{1 - q^2} \\
&= 2q^u p^2 \cdot \frac{1}{(1 + q)p} \\
&= \frac{2q^u p}{1 + q},
\end{aligned}$$

and for $u = 0$,

$$\begin{aligned}
\mathrm{P}(U = 0) &= \mathrm{P}(X = Y) \\
&= \sum_{t=1}^{\infty} \mathrm{P}(X = Y = t) \\
&= \sum_{t=1}^{\infty} \mathrm{P}(X = t)\,\mathrm{P}(Y = t) \\
&= \sum_{t=1}^{\infty} q^{t-1} p q^{t-1} p \\
&= p^2 \sum_{t=1}^{\infty} q^{2t-2} \\
&= p^2 \cdot \frac{1}{1 - q^2} \\
&= \frac{p}{1 + q}.
\end{aligned}$$

Since $W = \min\{X, Y\}$, we have $W \geq 1$. For $w \geq 1$, we have

$$\begin{aligned}
\mathrm{P}(W = w) &= \mathrm{P}(X = Y = w) + \mathrm{P}(X = w, Y > X) + \mathrm{P}(Y = w, Y < X) \\
&= \mathrm{P}(X = w, Y = w) + 2\,\mathrm{P}(X = w, Y > X) \\
&= \mathrm{P}(X = w)\,\mathrm{P}(Y = w) + 2\sum_{r=w+1}^{\infty} \mathrm{P}(X = w, Y = r) \\
&= \mathrm{P}(X = w)\,\mathrm{P}(Y = w) + 2\sum_{r=w+1}^{\infty} \mathrm{P}(X = w)\,\mathrm{P}(Y = r) \\
&= q^{w-1}pq^{w-1}p + 2\sum_{r=w+1}^{\infty} q^{w-1}pq^{r-1}p \\
&= q^{2w-2}p^2 + 2q^{w-2}p^2 \sum_{r=w+1}^{\infty} q^r \\
&= q^{2w-2}p^2 + 2q^{w-2}p^2q^{w+1} \cdot \frac{1}{1-q} \\
&= q^{2w-2}p^2 + 2q^{2w-1}p^2 \cdot \frac{1}{p} \\
&= q^{2w-2}p^2 + 2q^{2w-1}p \\
&= q^{2w-2}p\,(p + 2q) \\
&= q^{2w-2}p\,(1 + q)\,.
\end{aligned}$$

3. Since $S = 2$ and $T = 3$, the maximum of X and Y is 3, but they sum to 2, and this is impossible, so

$$\mathrm{P}(S = 2, T = 3) = 0.$$

However,

$$\begin{aligned}
\mathrm{P}(S = 2)\,\mathrm{P}(T = 3) &= (2-1)q^{2-2}p^2 \cdot pq^{3-1}(2 + q^3 - q^{3-1}) \\
&= p^3q^2(2 + q^3 - q^2) \\
&\neq 0 \\
&= \mathrm{P}(S = 2, T = 3),
\end{aligned}$$

as desired.

4. • U and W are independent. We split this into two cases of U to consider:
   – When $U = 0$, $X = Y$, and hence $W = X = Y$. In this case,

   $$\mathrm{P}(U = 0, W = w) = \mathrm{P}(X = Y = w) = q^{2w-2}p^2$$

   and notice

   $$\mathrm{P}(U = 0)\,\mathrm{P}(W = w) = \frac{p}{1+q} \cdot q^{2w-2}p(1+q) = p^2q^{2w-2},$$

   so

   $$\mathrm{P}(U = 0, W = w) = \mathrm{P}(U = 0)\,\mathrm{P}(W = w).$$

   – When $U = u \neq 0$,

   $$\begin{aligned}
   \mathrm{P}(U = u, W = w) &= \mathrm{P}(X = w, Y = w + u) + \mathrm{P}(X = w + u, Y = w) \\
   &= 2\,\mathrm{P}(X = w, Y = w + u) \\
   &= 2\,\mathrm{P}(X = w)\,\mathrm{P}(Y = w + u) \\
   &= 2q^{w-1}pq^{w+u-1}p \\
   &= 2q^{2w+u-2}p^2,
   \end{aligned}$$

   and

   $$\mathrm{P}(U = u)\,\mathrm{P}(W = w) = \frac{2q^up}{1+q} \cdot q^{2w-2}p(1+q) = 2q^{2w+u-2}p^2,$$

   and so

   $$\mathrm{P}(U = u, W = w) = \mathrm{P}(U = u)\,\mathrm{P}(W = w).$$

Hence, we can see that U and W are independent.

- U and S are not independent. Consider the case where $S = 3$ and $U = 0$. The event $S = 3, U = 0$ is not possible since $S = X + Y$ and $U = |X - Y|$ must have the same odd-even parity, giving
$$\mathrm{P}(S = 3, U = 0) = 0.$$
On the other hand,
$$\mathrm{P}(S = 3)\,\mathrm{P}(U = 0) = 2qp^2 \cdot \frac{p}{1+q} = \frac{2qp^3}{1+p} \neq 0.$$
This means
$$\mathrm{P}(S = 3, U = 0) \neq \mathrm{P}(S = 3)\,\mathrm{P}(U = 0),$$
and hence U and S are not independent.

- U and T are not independent. Consider the case where $U = 1$ and $T = 1$. The event $U = 1, T = 1$ implies that $X = Y \pm 1$, and that the maximum of X and Y is 1, and hence $X = Y = 1$, which is impossible. Hence,
$$\mathrm{P}(U = 1, T = 1) = 0.$$
On the other hand,
$$\mathrm{P}(U = 1)\,\mathrm{P}(T = 1) = \frac{2qp}{1+q} \cdot p(2 + q - 1) = 2p^2q \neq 0.$$
This means
$$\mathrm{P}(U = 1, T = 1) \neq \mathrm{P}(U = 1)\,\mathrm{P}(T = 1),$$
and hence U and T are not independent.

- W and S are not independent. Consider the case where $W = 2$ and $S = 2$. On one hand, since $\min\{X, Y\} = 2$, and $S = X + Y = \max\{X, Y\} + \min\{X, Y\} = 2$, this means $\max\{X, Y\} = 0$ which is impossible, and hence
$$\mathrm{P}(W = 2, S = 2) = 0.$$
On the other hand,
$$\mathrm{P}(W = 2)\,\mathrm{P}(S = 2) = q^2p(1+q)(1)q^0p^2 = p^3q^2(1+q) \neq 0.$$
This means
$$\mathrm{P}(W = 2, S = 2) = \mathrm{P}(W = 2)\,\mathrm{P}(S = 2).$$

- W and T are not independent. Consider the case where $T = 1$ and $W = 2$. Since $T = \max\{X, Y\} = 1$ and $W = \min\{X, Y\} = 2$, the event $T = 1, W = 2$ is not possible, hence
$$\mathrm{P}(T = 1, W = 2) = 0.$$
On the other hand,
$$\mathrm{P}(T = 1)\,\mathrm{P}(W = 2) = p(2 + q - 1)q^2p(1+q) = p^2q^2(1+q)^2 \neq 0.$$
This means
$$\mathrm{P}(T = 1, W = 2) \neq \mathrm{P}(T = 1)\,\mathrm{P}(W = 2),$$
and hence W and T are not independent.

- S and T are not independent. Counter-example shown in the previous part.