2020.3 Question 11

1. Since $X \sim \mathrm{U}[a, b]$, we must have for the probability density function of X, that

$$f_X(x) = \frac{1}{b-a}$$

for $x \in [a, b]$, and 0 everywhere else. Hence, the cumulative distribution function of X is

$$F_X(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{b-a}, & a < x \leq b, \\ b, & \text{otherwise.} \end{cases}$$

Since f is bijective and strictly decreasing on $[a, b]$, we must have for $y \in [a, b]$, that

$$\begin{aligned} \mathrm{P}(Y \leq y) &= \mathrm{P}(f(X) \leq y) \\ &= \mathrm{P}(X \geq f^{-1}(y)) \\ &= \mathrm{P}(X \geq f(y)) \\ &= 1 - \mathrm{P}(X < f(y)) \\ &= 1 - F_X(f(y)) \\ &= 1 - \frac{f(y) - a}{b-a} \\ &= \frac{(b-a) - (f(y) - a)}{b-a} \\ &= \frac{b - f(y)}{b-a}, \end{aligned}$$

as desired.

Hence, by differentiation with respect to y, we have the probability density function of Y satisfies

$$f_Y(y) = -\frac{f'(y)}{b-a}.$$

Hence, by the definition of expectation, we have

$$\begin{aligned} \mathrm{E}(y^2) &= \int_a^b f_Y(y) y^2 \, \mathrm{d}y \\ &= -\frac{1}{b-a} \int_a^b -f'(y) y^2 \, \mathrm{d}y \\ &= -\frac{1}{b-a} \int_a^b y^2 \, \mathrm{d}f(y) \\ &= \frac{1}{b-a} \left[- \left[y^2 f(y) \right]_a^b - 2 \int_a^b y f(y) \, \mathrm{d}Y \right] \\ &= \frac{1}{b-a} \left[-b^2 f(b) + a^2 f(a) + 2 \int_a^b y f(y) \, \mathrm{d}Y \right] \\ &= \frac{1}{b-a} \left[\frac{b}{3}(b^3 - a^3) - b^2 a + a^2 b + 2 \int_a^b y f(y) \, \mathrm{d}x \right] \\ &= \frac{b}{3} \left(b^2 + ab + a^2 \right) - ab + \int_a^b \frac{2x f(x) \, \mathrm{d}x}{b-a}. \end{aligned}$$

2. Since $\frac{1}{Z} + \frac{1}{X} = \frac{1}{c}$, by rearranging, we have

$$Z = \frac{1}{\frac{1}{c} - \frac{1}{X}} = \frac{cX}{X-c}.$$

By given, we have
$$c = \frac{ab}{a+b},$$
and hence
$$c < a, c < b.$$
Let $f(x) = \frac{cx}{x-c}$. Notice that
$$\begin{aligned}
f(a) &= \frac{ac}{a-c} \\
&= \frac{a^2b/(a+b)}{a-ab/(a+b)} \\
&= \frac{a^2b}{a^2+ab-ab} \\
&= b,
\end{aligned}$$
and
$$\begin{aligned}
f(b) &= \frac{bc}{b-c} \\
&= \frac{ab^2/(a+b)}{b-ab/(a+b)} \\
&= \frac{ab^2}{b^2+ab-ab} \\
&= a.
\end{aligned}$$
Also, since
$$f(x) = \frac{1}{\frac{x-c}{cx}} = \frac{1}{\frac{1}{c}-\frac{1}{x}},$$
as x strictly increases, $\frac{1}{x}$ strictly decreases, $-\frac{1}{x}$ strictly increases, the denominator strictly increases, and hence $f(x)$ strictly decreases.

Note that
$$\frac{1}{f(x)} + \frac{1}{x} = \frac{1}{c},$$
and hence
$$\frac{1}{x} + \frac{1}{f^{-1}(x)} = \frac{1}{c},$$
which implies
$$f(x) = f^{-1}(x).$$
So $Z = f(X)$ for this f satisfying all three conditions above. Hence,
$$\begin{aligned}
\mathrm{E}(Z) &= \int_a^b f(x) f_X(x) \,\mathrm{d}x \\
&= \frac{1}{b-a} \int_a^b \frac{cx \,\mathrm{d}x}{x-c} \\
&= \frac{1}{b-a} \int_a^b \left(c + \frac{c^2}{x-c}\right) \mathrm{d}x \\
&= \frac{1}{b-a} \left[cx + c^2 \ln|x-c|\right]_a^b \\
&= \frac{1}{b-a} \left[\left(cb + c^2 \ln|b-c|\right) - \left(ca + c^2 \ln|a-c|\right)\right] \\
&= c + \frac{c^2}{b-a} \ln\left|\frac{b-c}{a-c}\right| \\
&= c + \frac{c^2}{b-a} \ln\left(\frac{b-c}{a-c}\right),
\end{aligned}$$

and using the result from the previous part,

$$\begin{aligned}
\mathrm{E}(Z^2) &= -ab + \int_a^b \frac{2xf(x)}{b-a}\,\mathrm{d}x \\
&= -ab + \frac{2}{b-a} \cdot \int_a^b \frac{cx^2}{x-c}\,\mathrm{d}x \\
&= -ab + \frac{2c}{b-a} \cdot \int_a^b \left(x + c + \frac{c^2}{x-c}\right)\mathrm{d}x \\
&= -ab + \frac{2c}{b-a} \cdot \left[\frac{x^2}{2} + cx + c^2 \ln|x-c|\right]_a^b \\
&= -ab + \frac{2c}{b-a} \cdot \left[\left(\frac{b^2}{2} + bc + c^2 \ln|b-c|\right) - \left(\frac{a^2}{2} + ac + c^2 \ln|a-c|\right)\right] \\
&= -ab + \frac{2c}{b-a} \cdot \left[(b-a)\left(c + \frac{a+b}{2}\right) + c^2 \ln\left|\frac{b-c}{a-c}\right|\right] \\
&= -ab + 2c\left(c + \frac{a+b}{2}\right) + \frac{2c^3}{b-a} \ln\left(\frac{b-c}{a-c}\right) \\
&= 2c^2 + (a+b)c - ab + \frac{2c^3}{b-a} \ln\left(\frac{b-c}{a-c}\right) \\
&= 2c^2 + \frac{2c^3}{b-a} \ln\left(\frac{b-c}{a-c}\right).
\end{aligned}$$

Hence, the variance of Z satisfies that

$$\begin{aligned}
\mathrm{Var}(Z) &= \mathrm{E}(Z^2) - \mathrm{E}(Z)^2 \\
&= 2c^2 + \frac{2c^3}{b-a} \ln\left(\frac{b-c}{a-c}\right) - \left(c + \frac{c^2}{b-a} \ln\left(\frac{b-c}{a-c}\right)\right) \\
&= 2c^2 + \frac{2c^3}{b-a} \ln\left(\frac{b-c}{a-c}\right) - c^2 - \frac{2c^3}{b-a} \ln\left(\frac{b-c}{a-c}\right) - \frac{c^4}{(b-a)^2} \ln\left(\frac{b-c}{a-c}\right)^2 \\
&= c^2 - \frac{c^4}{(b-a)^2} \ln\left(\frac{b-c}{a-c}\right)^2.
\end{aligned}$$

Therefore, since the variance of a non-constant random variable is always positive,

$$\begin{aligned}
c^2 &> \frac{c^4}{(b-a)^2} \ln\left(\frac{b-c}{a-c}\right)^2 \\
(b-a)^2 &> c^2 \ln\left(\frac{b-c}{a-c}\right)^2 \\
|b-a| &> \left|c \ln\left(\frac{b-c}{a-c}\right)\right| \\
\left|\ln\left(\frac{b-c}{a-c}\right)\right| &< \left|\frac{b-a}{c}\right|.
\end{aligned}$$

Notice that since $b > a$, we must have $b - c > a - c$, so the natural log on the left-hand side is positive, and the fraction within the absolute value on the right-hand side is positive as well, and hence

$$\ln\left(\frac{b-c}{a-c}\right) < \frac{b-a}{c}.$$