2020.3 Question 1

1. Using integration by parts, we have

$$\begin{aligned}
I(a,b) &= \int_0^{\frac{\pi}{2}} \cos^a x \cos bx \,\mathrm{d}x \\
&= \frac{1}{b} \int_0^{\frac{\pi}{2}} \cos^a x \,\mathrm{d} \sin bx \\
&= \frac{1}{b} \left[(\cos^a x \sin bx)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin bx \,\mathrm{d} \cos^a x \right] \\
&= -\frac{1}{b} \int_0^{\frac{\pi}{2}} \sin bx \,\mathrm{d} \cos^a x \\
&= \frac{a}{b} \int_0^{\frac{\pi}{2}} \sin bx \sin x \cos^{a-1} x \,\mathrm{d}x.
\end{aligned}$$

Notice that

$$\cos(b-1)x = \cos bx \cos x + \sin bx \sin x,$$

and hence

$$\begin{aligned}
I(a-1,b-1) &= \int_0^{\frac{\pi}{2}} \cos^{a-1} x \cos(b-1)x \,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \cos^{a-1} x (\cos bx \cos x + \sin bx \sin x) \,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \cos^a x \cos bx \,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \sin bx \sin x \cos^{a-1} x \,\mathrm{d}x \\
&= I(a,b) + \frac{b}{a} I(a,b) \\
&= \frac{a+b}{a} I(a,b),
\end{aligned}$$

and hence

$$I(a,b) = \frac{a}{a+b} I(a-1,b-1),$$

as desired.

2. We look at the base case where $n=0$, and we have

$$\begin{aligned}
\text{LHS} &= \int_0^{\frac{\pi}{2}} \cos(2m+1)x \,\mathrm{d}x \\
&= \frac{1}{2m+1} \left[\sin(2m+1)x \right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2m+1} \sin \frac{(2m+1)\pi}{2} \\
&= \frac{(-1)^m}{2m+1},
\end{aligned}$$

and

$$\text{RHS} = (-1)^m \frac{2^0 0!(2m)!m!}{m!(2m+1)!} = \frac{(-1)^m}{2m+1},$$

and so LHS = RHS, which means this holds for the base case where $n=0$.

Now assume this is true for some $n = k \geq 0$, i.e.

$$I(k, 2m+k+1) = (-1)^m \frac{2^k k!(2m)!(k+m)!}{m!(2k+2m+1)!},$$

and we look at the case $n = k + 1$. Note that

$$\begin{aligned}
\text{LHS} &= I(k+1, 2m+k+2) \\
&= \frac{k+1}{2m+2k+3} I(k, 2m+k+1) \\
&= \frac{k+1}{2m+2k+3} (-1)^m \frac{2^k k!(2m)!(k+m)!}{m!(2k+2m+1)!} \\
&= (-1)^m \frac{2^k k!(2m)!(k+m)!(k+1)}{m!(2k+2m+1)!(2k+2m+3)} \\
&= (-1)^m \frac{2^k (k+1)!(2m)!(k+m)!2(k+m+1)}{m!(2k+2m+1)!(2k+2m+2)(2k+2m+3)} \\
&= (-1)^m \frac{2^{k+1}(k+1)!(2m)!(k+m+1)!}{m!(2k+2m+3)!} \\
&= (-1)^m \frac{2^{k+1}(k+1)!(2m)![(k+1)+m]!}{m!(2(k+1)+2m+1)!},
\end{aligned}$$

which shows the original statement is true for $n = k + 1$.

Hence, by the principle of mathematical induction, the original statement is true for any non-negative integers n, m.