2019.3 Question 8

1. W.L.O.G. let the origin be the centre of the rectangle $ABCD$ (and let $ABCD$ lie on the x-y plane). We adjust the scale of the axis, and we let $V(0,0,1)$ and $A(-\mu,-\nu,0)$, we have $B(\mu,-\nu,0)$, $C(\mu,\nu,0)$ and $D(-\mu,\nu,0)$. Let $\mu,\nu>0$.

   Let M be the midpoint of AB and N be the midpoint of BC. We must have $M(0,-\nu,0)$ and $N(\mu,0,0)$.

   The angle between the face AVB and the base $ABCD$ must be the angle between $\overrightarrow{MO}$ and $\overrightarrow{MV}$. Hence,

$$\cos\alpha=\frac{\overrightarrow{MO}\cdot\overrightarrow{MV}}{\left|\overrightarrow{MO}\right|\left|\overrightarrow{MV}\right|}.$$

   Note that

$$\overrightarrow{MO}=\begin{pmatrix}0\\\nu\\0\end{pmatrix},\overrightarrow{MV}=\mathbf{v}-\mathbf{m}=\begin{pmatrix}0\\\nu\\1\end{pmatrix},$$

   and hence

$$\cos\alpha=\frac{\nu^2}{\nu\cdot\sqrt{\nu^2+1}}=\frac{\nu}{\sqrt{\nu^2+1}},$$

   which gives

$$\cos^2\alpha\nu^2+\cos^2\alpha=\nu^2,$$

   and hence

$$\sin^2\alpha\nu^2=\cos^2\alpha,$$

   which gives

$$\nu=\cot\alpha.$$

   Similarly,

$$\mu=\cot\beta.$$

   A vector perpendicular to AVB can be

$$\begin{aligned}\overrightarrow{VA}\times\overrightarrow{VB}&=\begin{pmatrix}-\mu\\-\nu\\-1\end{pmatrix}\times\begin{pmatrix}\mu\\-\nu\\-1\end{pmatrix}\\&=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\-\mu&-\nu&-1\\\mu&-\nu&-1\end{vmatrix}\\&=\begin{pmatrix}0\\-2\mu\\2\mu\nu\end{pmatrix}\\&=\begin{pmatrix}0\\-2\cot\beta\\2\cot\alpha\cot\beta.\end{pmatrix}\\&=-\frac{2\cot\beta}{\sin\alpha}\begin{pmatrix}0\\-\sin\alpha\\\cos\alpha\end{pmatrix},\end{aligned}$$

   and so

$$\begin{pmatrix}0\\-\sin\alpha\\\cos\alpha\end{pmatrix}$$

   is a unit vector perpendicular to AVB.

Similarly,

$$\begin{aligned}\overrightarrow{VB} \times \overrightarrow{VC} &= \begin{pmatrix}\mu \\ -\nu \\ -1\end{pmatrix} \times \begin{pmatrix}\mu \\ \nu \\ -1\end{pmatrix} \\ &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \mu & -\nu & -1 \\ \mu & \nu & -1\end{vmatrix} \\ &= \begin{pmatrix}2\nu \\ 0 \\ 2\mu\nu\end{pmatrix} \\ &= \begin{pmatrix}2\cot\alpha \\ 0 \\ 2\cot\alpha\cot\beta\end{pmatrix} \\ &= \frac{2\cot\alpha}{\sin\beta}\begin{pmatrix}\sin\beta \\ 0 \\ \cos\beta\end{pmatrix},\end{aligned}$$

and hence

$$\begin{pmatrix}\sin\beta \\ 0 \\ \cos\beta\end{pmatrix}$$

is a unit vector perpendicular to BVC.

The acute angle between AVB and BVC, θ, satisfies that

$$\cos\theta = \begin{pmatrix}0 \\ -\sin\alpha \\ \cos\alpha\end{pmatrix} \cdot \begin{pmatrix}\sin\beta \\ 0 \\ \cos\beta\end{pmatrix} = \cos\alpha\cos\beta,$$

as desired.

2. Notice that

$$\begin{aligned}\cos\varphi &= \frac{\overrightarrow{BV} \cdot \overrightarrow{BO}}{\left|\overrightarrow{BV}\right| \cdot \left|\overrightarrow{BO}\right|} \\ &= \frac{\begin{pmatrix}-\mu \\ \nu \\ 1\end{pmatrix} \cdot \begin{pmatrix}-\mu \\ \nu \\ 0\end{pmatrix}}{\sqrt{\mu^2+\nu^2+1}\sqrt{\mu^2+\nu^2}} \\ &= \sqrt{\frac{\mu^2+\nu^2}{\mu^2+\nu^2+1}},\end{aligned}$$

and hence

$$\sin\varphi = \sqrt{1-\cos^2\varphi} = \sqrt{\frac{1}{\mu^2+\nu^2+1}},$$

which means

$$\cot\varphi = \sqrt{\mu^2+\nu^2},$$

and hence

$$\cot^2\varphi = \mu^2+\nu^2 = \cot^2\alpha + \cot^2\beta,$$

as desired.

Notice that

$$\begin{aligned}\cos^2\varphi &= \frac{\mu^2+\nu^2}{\mu^2+\nu^2+1}\\ &= \frac{\cot^2\alpha+\cot^2\beta}{\cot^2\alpha+\cot^2\beta+1}\\ &= \frac{\cos^2\alpha\sin^2\beta+\cos^2\beta\sin^2\alpha}{\cos^2\alpha\sin^2\beta+\cos^2\beta\sin^2\alpha+\sin^2\beta\sin^2\alpha}\\ &= \frac{\cos^2\alpha(1-\cos^2\beta)+\cos^2\beta(1-\cos^2\alpha)}{(\cos^2\alpha+\sin^2\alpha)(\cos^2\beta+\sin^2\beta)-\cos^2\alpha\cos^2\beta}\\ &= \frac{\cos^2\alpha+\cos^2\beta-2\cos^2\alpha\cos^2\beta}{1-\cos^2\alpha\cos^2\beta}\\ &= \frac{\cos^2\alpha+\cos^2\beta-2\cos^2\theta}{1-\cos^2\theta}.\end{aligned}$$

Since $(\cos\alpha-\cos\beta)^2=\cos^2\alpha+\cos^2\beta-2\cos\theta\geq 0$, we have $\cos^2\alpha+\cos^2\beta\geq 2\cos\theta$, and hence

$$\cos^2\varphi=\frac{\cos^2\alpha+\cos^2\beta-2\cos^2\theta}{1-\cos^2\theta}\geq\frac{2\cos\theta-2\cos^2\theta}{1-\cos^2\theta}.$$

Notice that

$$\begin{aligned}\cos^2\varphi &\geq \frac{2\cos\theta-2\cos^2\theta}{1-\cos^2\theta}\\ &= \frac{2\cos\theta(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}\\ &= \frac{2\cos\theta}{1+\cos\theta}\\ &= \frac{2}{1+\cos\theta}\cos\theta\\ &> \frac{2}{1+1}\cos\theta\\ &= \cos\theta\\ &> \cos^2\theta,\end{aligned}$$

since θ is acute, $0<\cos\theta<1$.

This means $\cos^2\varphi>\cos^2\theta$, and since θ,φ are acute, this must mean that $\varphi<\theta$, since $\cos\varphi,\cos\theta$ are both positive, and $\cos\varphi>\cos\theta$.