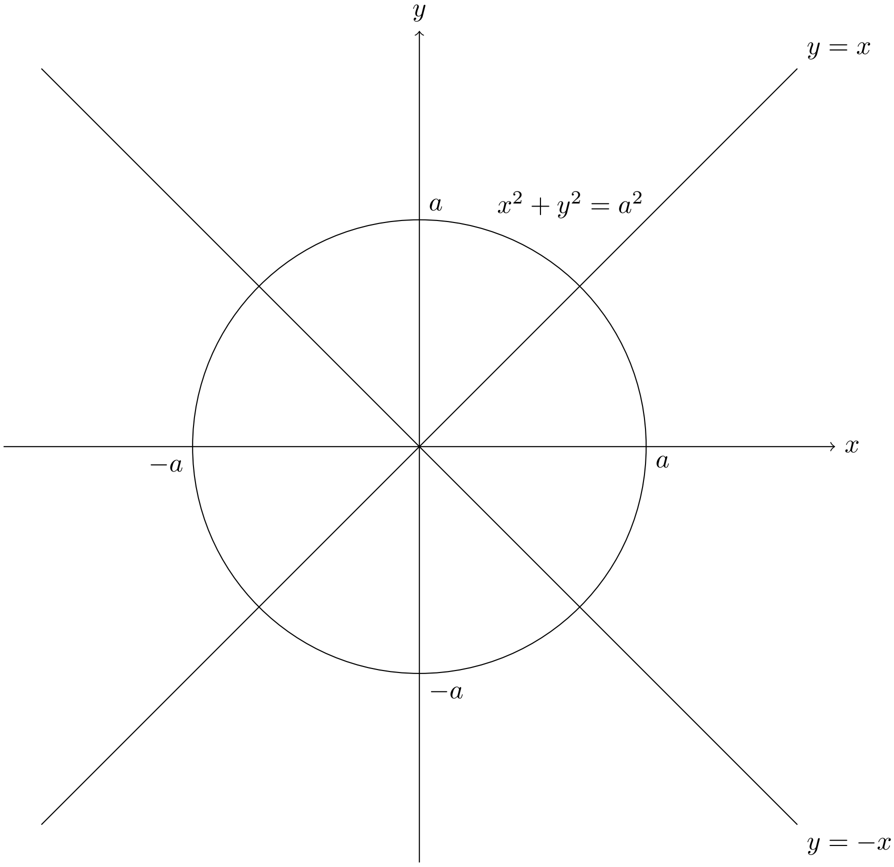
2019.3 Question 7

1. When $a = b$,

$$\begin{aligned} y^2(y^2 - a^2) &= x^2(x^2 - a^2) \\ x^4 - y^4 - a^2x^2 + a^2y^2 &= 0 \\ (x^2 + y^2 - a^2)(x^2 - y^2) &= 0 \\ (x^2 + y^2 - a^2)(x + y)(x - y) &= 0, \end{aligned}$$

so the Devil's Curve in this case consists of the line $x + y = 0$, the line $x - y = 0$, and the circle $x^2 + y^2 = a^2$.

The curve is shown as follows.



2. When $a = 2$ and $b = \sqrt{5}$,

$$y^2(y^2 - 5) = x^2(x^2 - 4).$$

(a) Rearrangement gives us

$$(x^2)^2 - 4x^2 - y^2(y^2 - 5) = 0,$$

and considering the discriminant, we have

$$(-4)^2 + 4y^2(y^2 - 5) \geq 0,$$

i.e.

$$\left(y^2 - 1\right)\left(y^2 - 4\right) \geq 0.$$

This gives $y^2 \leq 1$ or $y^2 \geq 4$, and in the case where $y \geq 0$, this must give $0 \leq y \leq 1$ or $y \geq 2$, as desired.

(b) When the curve is very close to the origin, we must have $x^4, y^4 \ll x^2, y^2$, and hence $4x^2 \approx 5y^2$, which means $y \approx \frac{2}{\sqrt{5}}x$.

When the curve is very far from the origin, we must have $x^4, y^4 \gg x^2, y^2$, and hence $x^4 \approx y^4$, which means $y \approx x$.

(c) Using implicit differentiation, we have

$$\begin{aligned} y^2(y^2-5) &= x^2(x^2-4) \\ (4y^3-10y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x^3-8x \\ (2y^2-5)y\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x(x^2-2). \end{aligned}$$

When $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, the tangent to the curve is parallel to the x-axis, and hence

$$2x(x^2-2)=0,$$

giving $x=0$ or $x=\sqrt{2}$.
For $x=0$, $y^2(y^2-5)=0$, and therefore $y=0$ or $y=\sqrt{5}$. The case where $y=0$ does not necessarily give that $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, but the case where $y=\sqrt{5}$ does.
For $x=\sqrt{2}$, $y^2(y^2-5)=-4$, $y=2$ or $y=1$. Both cases give $\frac{\mathrm{d}y}{\mathrm{d}x}=0$.
So the tangent to the curve is parallel to the x-axis at points

$$\left(0,\sqrt{5}\right),\left(\sqrt{2},1\right),\left(\sqrt{2},2\right).$$

We must have

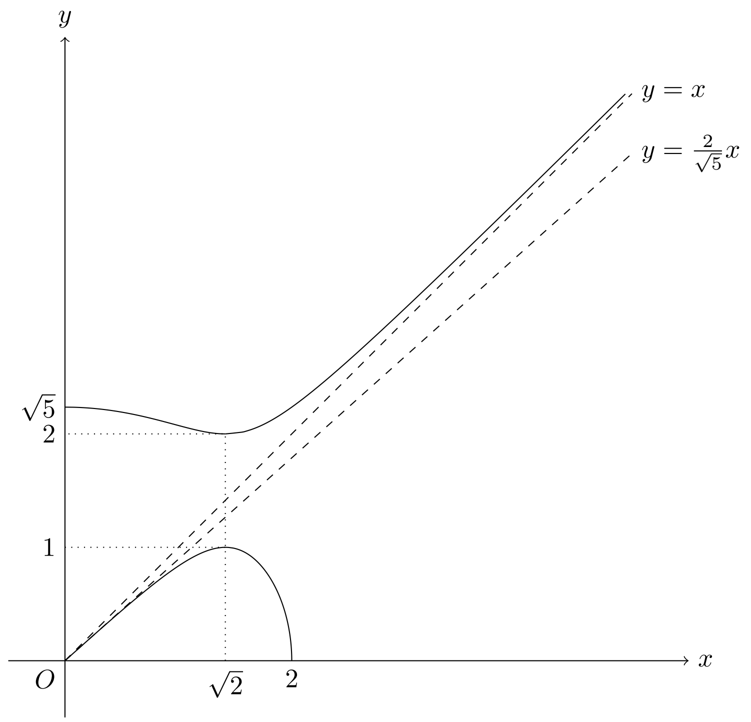
$$(2y^2-5)y=2x(x^2-2)\frac{\mathrm{d}x}{\mathrm{d}y},$$

and when $\frac{\mathrm{d}x}{\mathrm{d}y}=0$, the tangent to the curve is parallel to the y-axis.
This gives $(2y^2-5)y=0$, and hence $y=0$ or $y=\sqrt{\frac{5}{2}}$.
For $y=0$, $x=0$ or $x=2$. The case $x=0$ does not necessarily give $\frac{\mathrm{d}x}{\mathrm{d}y}=0$, but the case where $x=2$ does.
For $y=\sqrt{\frac{5}{2}}$, $x^2(x^2-4)=-\frac{25}{4}$, and hence

$$4x^4-16x^2+25=4(x^2-2)^2+9=0,$$

which is not possible.
Hence, the tangent to the curve is parallel to the y-axis only at $(2,0)$.

Therefore, from the analysis in the previous parts, the curve looks as follows for $x\geq 0$ and $y\geq 0$:



3. All x terms in the curve is in x^2, so the graph is symmetric in the y-axis since $x^2 = (-x)^2$. Similarly, the graph is symmetric in the x-axis as well. Hence, the complete graph looks as follows.