2019.3 Question 6

Notice that the original equation

$$zz^* - az^* - a^*z + aa^* - r^2 = 0$$

can be simplified to

$$(z-a)(z^*-a^*) = r^2,$$

and the left-hand side satisfies

$$(z-a)(z^*-a^*) = (z-a)(z-a)^* = |z-a|^2,$$

which means the original equation is

$$|z-a|^2 = r^2,$$

and hence

$$|z-a| = r.$$

This is a circle centred at a with radius r.

1. Since $\omega = \frac{1}{z}$, we have $z = \frac{1}{\omega}$. Hence,

$$\begin{aligned}
\frac{1}{\omega}\frac{1}{\omega^*} - a\frac{1}{\omega^*} - \frac{1}{\omega}a^* + aa^* &= r^2 \\
1 - \omega a - \omega^* a^* + aa^*\omega\omega^* &= r^2\omega\omega^* \\
(r^2 - aa^*)\omega\omega^* + \omega a + \omega^* a^* &= 1 \\
\omega\omega^* + \frac{a}{r^2-aa^*}\omega + \frac{a^*}{r^2-aa^*}\omega^* &= \frac{1}{r^2-aa^*} \\
\left(\omega + \frac{a^*}{r^2-aa^*}\right)\left(\omega + \frac{a^*}{r^2-aa^*}\right)^* &= \frac{1}{r^2-aa^*} + \frac{aa*}{(r^2-aa^*)^2} \\
\left|\omega - \frac{a^*}{aa^*-r^2}\right|^2 &= \frac{r^2}{(r^2-aa^*)^2} \\
\left|\omega - \frac{a^*}{aa^*-r^2}\right| &= \frac{r}{|r^2-aa^*|},
\end{aligned}$$

so ω is on a circle C' with centre $\frac{a^*}{aa^*-r^2}$ and radius $\frac{r}{|r^2-aa^*|}$.

If C and C' are the same circle, we have

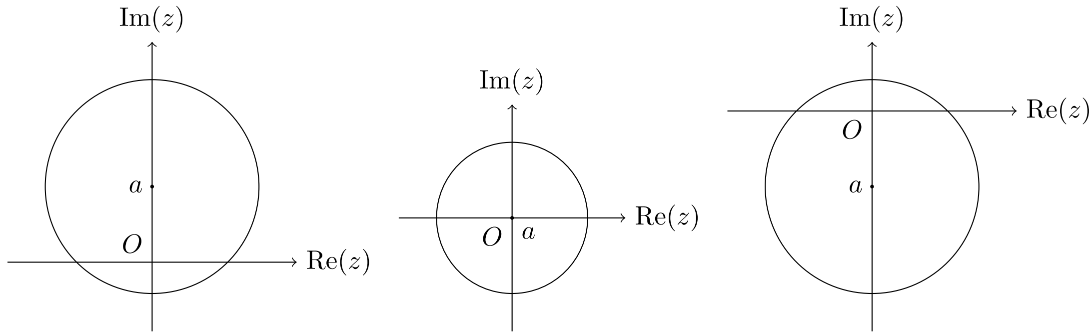
$$a = \frac{a^*}{aa^*-r^2}, r = \frac{r}{|r^2-aa^*|}.$$

The second equation gives $|r^2 - aa^*| = 1$, which means $r^2 - aa^* = \pm 1$.

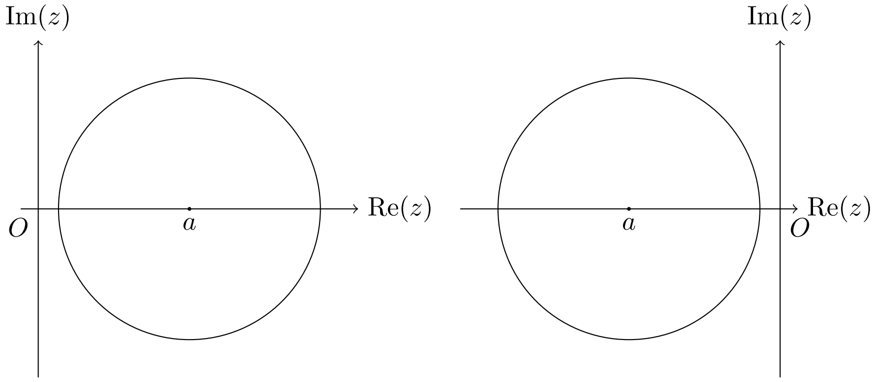
$$\begin{aligned}
r^2 - aa^* &= \pm 1 \\
r^2 - |a|^2 &= \pm 1 \\
\left(|a|^2 - r^2\right)^2 &= 1,
\end{aligned}$$

as desired.

When $r^2 - aa^* = 1$, $a = -a^*$, and hence a is pure imaginary. Since $r^2 = 1 + |a|^2$ in this case, $r > |a|$, so the circle must contain the origin. The diagrams are as below, with the case $\text{Im}(a) > 0$ on the left, $\text{Im}(a) = 0$ in the middle, and $\text{Im}(a) < 0$ on the right:



When $r^2 - aa^* = -1$, $a = a^*$, and hence a is real. Since $r^2 = -1 + |a|^2$ in this case, $r < |a|$, so the circle cannot contain the origin, and $|a| > 1$. The diagrams are as below, with the case $\text{Re}(a) > 1$ on the left, and $\text{Re}(a) < -1$ on the right:



2. In the case where $\omega = \frac{1}{z^*}$, we have $z = \frac{1}{\omega^*}$, and hence similar to the previous one,

$$\begin{aligned}
\omega\omega^* + \frac{a}{r^2 - aa^*}\omega^* + \frac{a^*}{r^2 - aa^*}\omega &= \frac{1}{r^2 - aa^*} \\
\left(\omega + \frac{a}{r^2 - aa^*}\right)\left(\omega + \frac{a}{r^2 - aa^*}\right)^* &= \frac{1}{r^2 - aa^*} + \frac{aa*}{(r^2 - aa^*)^2} \\
\left|\omega - \frac{a}{aa^* - r^2}\right|^2 &= \frac{r^2}{(r^2 - aa^*)^2} \\
\left|\omega - \frac{a}{aa^* - r^2}\right| &= \frac{r}{|r^2 - aa^*|},
\end{aligned}$$

so ω is on a circle C' with centre $\frac{a}{aa^* - r^2}$ and radius $\frac{r}{|r^2 - aa^*|}$.

If they are the same circle, we have

$$a = \frac{a}{aa^* - r^2}, r = \frac{r}{|r^2 - aa^*|}.$$

We still have $r^2 - aa^* = \pm 1$.

When $r^2 - aa^* = 1$, we have $a = -a$, and $a = 0$.

When $r^2 - aa^* = -1$, we have $a = a$, and a can be any complex number satisfying $|a| = \sqrt{r^2 + 1}$.

It is not the case that a is either real or pure imaginary.