2019.3 Question 5

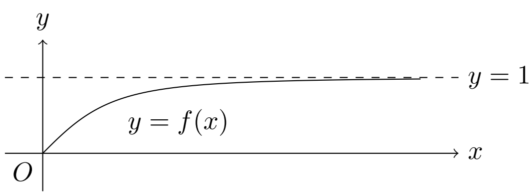
1. By quotient rule,

$$\begin{aligned} f'(x) &= \frac{\sqrt{x^2+p} - x \cdot \frac{1}{2} \cdot 2x \cdot \frac{1}{\sqrt{x^2+p}}}{x^2+p} \\ &= \frac{\sqrt{x^2+p} - \frac{x^2}{\sqrt{x^2+p}}}{x^2+p} \\ &= \frac{p}{(x^2+p)\sqrt{x^2+p}}. \end{aligned}$$

   This gives

$$0 < f'(x) \leq \frac{1}{\sqrt{p}},$$

   with the equal sign taking if and only if $x = 0$.
   $\lim_{x\to\infty} f(x) = 1$, so $y = 1$ is a horizontal asymptote to the function.
   Hence, the graph looks as follows:



2. Since $y = \frac{cx}{\sqrt{x^2+p}} = cf(x)$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = cf'(x) = \frac{cp}{\left(\sqrt{x^2+p}\right)^3},$$

   and hence

$$\mathrm{d}y = \frac{cp}{\left(\sqrt{x^2+p}\right)^3}\,\mathrm{d}x.$$

   The integral can therefore be simplified as

$$\begin{aligned} I &= \int \frac{\mathrm{d}y}{(b^2-y^2)\sqrt{c^2-y^2}} \\ &= \int \frac{1}{\left(b^2 - \frac{c^2x^2}{x^2+p}\right)\sqrt{c^2 - \frac{c^2x^2}{x^2+p}}} \cdot \frac{cp}{\left(\sqrt{x^2+p}\right)^3}\,\mathrm{d}x \\ &= \int \frac{cp\,\mathrm{d}x}{\left(b^2(x^2+p) - c^2x^2\right)\sqrt{c^2(x^2+p) - c^2x^2}} \\ &= \int \frac{cp\,\mathrm{d}x}{\left[(b^2-c^2)x^2 + b^2p\right]\sqrt{c^2p}} \\ &= \int \frac{\sqrt{p}\,\mathrm{d}x}{b^2p + (b^2-c^2)x^2}. \end{aligned}$$

   Let $p = 1$, and we have

$$I = \int \frac{\mathrm{d}x}{b^2 + (b^2-c^2)x^2}$$

   as desired.

Hence,

$$\begin{aligned}I &= \int \frac{\mathrm{d}x}{b^2+(b^2-c^2)x^2} \\ &= \frac{1}{b^2-c^2}\int \frac{\mathrm{d}x}{\left(\frac{b}{\sqrt{b^2-c^2}}\right)^2+x^2} \\ &= \frac{1}{b^2-c^2}\cdot\frac{\sqrt{b^2-c^2}}{b}\arctan\frac{\sqrt{b^2-c^2}x}{b}+C \\ &= \frac{1}{b\sqrt{b^2-c^2}}\arctan\frac{\sqrt{b^2-c^2}x}{b}+C.\end{aligned}$$

Let $b=\sqrt{3}$ and $c=\sqrt{2}$, and hence

$$I=\frac{1}{\sqrt{3}\sqrt{3-2}}\arctan\frac{\sqrt{3-2}x}{\sqrt{3}}+C=\frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}}+C.$$

When $y=1$, $\frac{\sqrt{2}x}{\sqrt{x^2+1}}=1$, and hence $x^2+1=2x^2$, $x^2=1$, giving $x=1$.
When $y\to\sqrt{2}=b$, $x\to\infty$.
Hence,

$$\int_1^{\sqrt{2}}\frac{\mathrm{d}y}{(3-y^2)\sqrt{2-y^2}}=\frac{1}{\sqrt{3}}\left[\arctan\frac{x}{\sqrt{3}}\right]_1^{\infty}=\frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\frac{\pi}{3\sqrt{3}}.$$

Consider letting $x=\frac{1}{y}$ in the integral, and we have $\mathrm{d}x=-\frac{1}{y^2}\,\mathrm{d}y=-x^2\,\mathrm{d}y$, and when $y=1$, $x=1$, and when $y=\sqrt{2}$, $x=\frac{1}{\sqrt{2}}$. Hence,

$$\begin{aligned}\int_{\frac{1}{\sqrt{2}}}^1\frac{y\,\mathrm{d}y}{(3y^2-1)\sqrt{2y^2-1}} &= \int_{\sqrt{2}}^1\frac{\frac{1}{x}\cdot\frac{1}{-x^2}\,\mathrm{d}x}{\left(\frac{3}{x^2}-1\right)\sqrt{\frac{2}{x^2}-1}} \\ &= \int_1^{\sqrt{2}}\frac{\mathrm{d}x}{(3-x^2)\sqrt{2-x^2}} \\ &= \frac{\pi}{3\sqrt{3}}.\end{aligned}$$

3. Consider the same substitution $y=\frac{ax}{\sqrt{x^2+p}}$. We still have

$$\mathrm{d}y=\frac{ap}{\left(\sqrt{x^2+p}\right)^3}\,\mathrm{d}x,$$

and hence

$$\begin{aligned}&\int\frac{\mathrm{d}y}{(3y^2-1)\sqrt{2y^2-1}} \\ &= \int\frac{ap}{\left(\sqrt{x^2+p}\right)^3}\cdot\frac{\mathrm{d}x}{\left(3\cdot\frac{a^2x^2}{x^2+p}-1\right)\sqrt{2\cdot\frac{a^2x^2}{x^2+p}-1}} \\ &= \int\frac{ap\,\mathrm{d}x}{(3a^2x^2-(x^2+p))\sqrt{2a^2x^2-(x^2+p)}} \\ &= \int\frac{ap\,\mathrm{d}x}{((3a^2-1)x^2-p)\sqrt{(2a^2-1)x^2-p}}.\end{aligned}$$

Consider letting $a = \frac{1}{\sqrt{2}}$ and $p = -1$, and we have

$$\begin{aligned}
&\int \frac{\mathrm{d}y}{(3y^2-1)\sqrt{2y^2-1}} \\
&= \int \frac{-\mathrm{d}x}{\sqrt{2}\left(\frac{1}{2}x^2+1\right)} \\
&= \int \frac{-\sqrt{2}\,\mathrm{d}x}{x^2+2} \\
&= -\sqrt{2}\cdot\frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + C \\
&= -\arctan\frac{x}{\sqrt{2}} + C.
\end{aligned}$$

When $y = \frac{1}{\sqrt{2}}$, we have $\frac{1}{\sqrt{2}} \cdot \frac{x}{\sqrt{x^2-1}} = \frac{1}{\sqrt{2}}$, and $x \to \infty$. When $y = 1$, we have $\frac{1}{\sqrt{2}} \cdot \frac{x}{\sqrt{x^2-1}} = 1$, and $x = \sqrt{2}$. Hence,

$$\begin{aligned}
&\int_{\frac{1}{\sqrt{2}}}^{1} \frac{\mathrm{d}y}{(3y^2-1)\sqrt{2y^2-1}} \\
&= -\left[\arctan\frac{x}{\sqrt{2}}\right]_{\infty}^{\sqrt{2}} \\
&= \left[\arctan\frac{x}{\sqrt{2}}\right]_{\sqrt{2}}^{\infty} \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4}.
\end{aligned}$$