2019.3 Question 4

1. We look at different cases depending on the value of n.

   - When $n = 1$, $P(x) = x - a_1$ has root a_1, and thus is reflective for all $a_1 \in \mathbb{R}$.
   - When $n = 2$, $P(x) = x^2 - a_1 x + a_2$ has root a_1, a_2, and hence by Vieta's Theorem,

     $$a_1 a_2 = a_2, a_1 + a_2 = a_1.$$

     This means $a_2 = 0$ and a_1 can take any real value, and hence

     $$P(x) = x^2 - a_1 x$$

     is reflective for $a_1 \in \mathbb{R}$.
   - When $n = 3$, $P(x) = x^3 - a_1 x^2 + a_2 x - a_3$ has root a_1, a_2, a_3, and hence by Vieta's Theorem,

     $$\begin{cases} a_1 a_2 a_3 = a_3, \\ a_1 a_2 + a_1 a_3 + a_2 a_3 = a_2, \\ a_1 + a_2 + a_3 = a_1. \end{cases}$$

     The final equation implies that $a_2 + a_3 = 0$, and hence with the second equation gives that $a_2 a_3 = a_2$, which means either $a_2 = a_3 = 0$, or $a_2 = -1, a_3 = 1$.
     When $a_2 = a_3 = 0$, a_1 can take any real value, and when $a_2 = -1, a_3 = 1$, we must have $a_1 = -1$.
     So the degree 3 reflective polynomials are

     $$P(x) = x^3 - a_1 x^2$$

     for all $a_1 \in \mathbb{R}$, and

     $$P(x) = x^3 + x^2 - x - 1.$$

2. By Vieta's Theorem, we have

   $$\sum_{i=1}^{n} a_i = a_1,$$

   and hence

   $$\sum_{i=2}^{n} a_i = 0.$$

   Squaring both sides gives

   $$\begin{aligned} 0 &= \left(\sum_{i=2}^{n} a_i\right)^2 \\ &= \sum_{i=2}^{n} a_i^2 + 2 \sum_{i=2}^{n-1} \sum_{j=i+1}^{n} a_i a_j. \end{aligned}$$

   By Vieta's Theorem, we also have

   $$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_i a_j = a_2,$$

and notice that

$$\begin{aligned}
2a_2 &= 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i a_j \\
&= 2\sum_{j=2}^{n} a_1 a_j + 2\sum_{i=2}^{n-1}\sum_{j=i+1}^{n} a_i a_j \\
&= 2a_1 \sum_{i=2}^{n} a_i + \left(-\sum_{i=2}^{n} a_i^2\right) \\
&= 2a_1 \cdot 0 - \sum_{i=2}^{n} a_i^2 \\
&= -\sum_{i=2}^{n} a_i^2,
\end{aligned}$$

as desired.

For the final part, assume B.W.O.C. that $n > 3$. By rearrangement, we have

$$a_2^2 + 2a_2 + 1 = 1 - \sum_{i=3}^{n} a_i^2,$$

and the left-hand side is $(a_2 + 1)^2$ which is always non-negative. Hence,

$$\sum_{i=3}^{n} a_i^2 \leq 1.$$

Since a_i are all integers, precisely one of the a_is for $3 \leq i \leq n$ is ± 1, and all the rest are 0. Since $a_n \neq 0$, we conclude that $a_n = \pm 1$, and $a_3 = \cdots = a_{n-1} = 0$.

But notice from Vieta's Theorem that

$$a_n = \prod_{i=1}^{n} a_i = 0$$

since a_3 must be 0, which leads to a contradiction.

Hence, we must have $n \leq 3$.

3. The reflective polynomials for $n \leq 3$ are
   - $P(x) = x - a_1$ for $a_1 \in \mathbb{Z}$,
   - $P(x) = x^2 - a_1 x$ for $a_1 \in \mathbb{Z}$,
   - $P(x) = x^3 - a_1 x^2$ for $a_1 \in \mathbb{Z}$, and
   - $P(x) = x^3 + x^2 - x - 1$.

   For $n > 3$, we must have $a_n = 0$, and hence

   $$\begin{aligned}
   P(x) &= x^n - a_1 x^{n-1} + a_2 x^{n-2} - \cdots + (-1)^{n-1} a_{n-1} x \\
   &= x\left(x^{n-1} - a_2 x^{n-2} + a_2 x^{n-3} - \cdots + (-1)^{n-1} a_{n-1}\right).
   \end{aligned}$$

   Let

   $$Q(x) = x^{n-1} - a_2 x^{n-2} + a_2 x^{n-3} - \cdots + (-1)^{n-1} a_{n-1}$$

   If $P(x)$ is reflective, then the roots to $P(x)$ are $a_1, a_2, \ldots, a_{n-1}, 0$, and hence the roots to $Q(x)$ are $a_1, a_2, \ldots, a_{n-1}$, which shows that $Q(x)$ is reflective as well.

   This means that an integer-coefficient reflective polynomial with degree $n > 3$ must be x multiplied by another integer-coefficient reflective polynomial, and repeating this process, we can conclude it must be some power of x multiplied by some integer-coefficient reflective polynomial with degree $n \leq 3$.

   Hence, all integer-coefficient reflective polynomials are

- $P(x) = x^r(x - a_1)$ for $a_1 \in \mathbb{Z}$, $r \in \mathbb{Z}$, $r \geq 0$, and
- $P(x) = x^r(x^3 + x^2 - x - 1) = x^2(x+1)^2(x-1)$ for $r \in \mathbb{Z}$, $r \geq 0$.