2019.3 Question 3

1. Since L_1 is a line of invariant points, for each point $(x, y) \in L_1$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

and hence

$$ax + by = x, cx + dy = y.$$

Hence,

$$(1 - a)x = by, (1 - d)y = cx,$$

and hence

$$(1 - a)x(1 - d)y = bycx,$$

which simplifies to

$$[(a - 1)(d - 1) - bc]xy = 0.$$

If the line L_1 is the line $x = 0$, then $by = 0$ for all y and $dy = y$ for all y, giving $d = 1$ and $b = 0$. Hence, $(a - 1)(d - 1) - bc = 0$.

Similarly, if the line L_1 is the line $y = 0$, then $ax = x$ for all x and $cx = 0$ for all y, giving $a = 1$ and $c = 0$. Hence, $(a - 1)(d - 1) - bc = 0$.

Otherwise, there must be a point $(x, y) \in L_1$ such that $xy \neq 0$, which means $(a-1)(d-1)-bc = 0$.

Hence, in all cases, we must have $(a - 1)(d - 1) = bc$ as desired.

If L_1 does not pass through the origin, then $y = mx + k$ for some $k \neq 0$, or $x = k$ for some $k \neq 0$.

In the first case, we have

$$ax + b(mx + k) = x,$$

and hence

$$(a + bm - 1)x + bk = 0$$

for all x, meaning $a + bm - 1 = 0$ and $bk = 0$.

Similarly,

$$cx + d(mx + k) = mx + k,$$

and hence

$$(c + dm - m)x + (d - 1)k = 0$$

for all x, meaning $c + dm - m = 0$ and $(d - 1)k = 0$.

Since $k \neq 0$, $bk = 0$ and $(d - 1)k = 0$ implies $b = 0$ and $d = 1$ respectively. Putting those back into the first corresponding equations, this solves to $a = 1$ and $c = 0$, which means

$$\mathbf{A} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \mathbf{I}_2.$$

In the second case where $x = k$ for some $k \neq 0$, we have

$$ak + by = k,$$

and hence

$$by + (a - 1)k = 0$$

for all y, meaning $b = 0$ and $(a - 1)k = 0$.

Similarly,

$$ck + dy = y,$$

and hence

$$(d - 1)y + ck = 0$$

for all y, meaning $d - 1 = 0$ and $ck = 0$.

Since $k \neq 0$, $(a-1)k = 0$ and $ck = 0$ implies $a = 1$ and $c = 0$ respectively. Hence,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}_2.$$

Therefore, L_1 not passing through the origin must imply that $\mathbf{A}$ is precisely the 2 by 2 identity matrix.

2. If (x, y) is an invariant point, we have

$$(a-1)x + by = 0, cx + (d-1)y = 0.$$

If $b = 0$, then $(a-1)(d-1) = bc = 0$, and hence $a = 1$ or $d = 1$.

In the case where $a = 1$, the first equation is trivially true, and the second equation simplifies to

$$cx + (d-1)y = 0,$$

and hence the line $L : cx + (d-1)y = 0$ is a line of invariant points.

In the case where $d = 1$, the original equation simplifies to

$$(a-1)x = 0, cx = 0,$$

and hence the line $L : x = 0$ is a line of invariant points.

If $b \neq 0$, we want to show that all points on the line $L : (a-1)x + by = 0$ satisfy the second equation. We multiply $(d-1)$ on both sides of the equation, and hence

$$(a-1)(d-1)x + b(d-1)y = 0,$$

which is

$$bcx + b(d-1)y = 0.$$

Since $b \neq 0$, we divide b on both sides, giving

$$cx + (d-1)y = 0,$$

which is precisely the second equation. Hence, $L : (a-1)x + by = 0$ is a line of invariant points under this case.

3. We have $L_2 : y = mx + k$, $k \neq 0$, we therefore have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} X \\ mX + k \end{pmatrix},$$

and hence

$$ax + b(mx + k) = X, cx + d(mx + k) = mX + k.$$

Putting the first equation into the second one gives us

$$cx + d(mx + k) = m(ax + b(mx + k)) + k,$$

which simplifies to

$$(c + dm - am - bm^2)x + (dk - mbk - k) = 0,$$

which is

$$(bm^2 + (a-d)m - c)x + (mb - d + 1)k = 0.$$

Since this is true for all x and $k \neq 0$, we must have

$$bm^2 + (a-d)m - c = 0, bm - d + 1 = 0.$$

If $b = 0$, then

$$(a-d)m = c, d - 1 = 0,$$

and hence $d = 1$, $(a-1)m = c$, and

$$(a-1)(d-1) = 0, bc = 0,$$

which gives

$$(a-1)(d-1) = bc.$$

If $b \neq 0$, the second of those equations solve to

$$m = \frac{d-1}{b},$$

and putting this back into the first equation, we have

$$b \cdot \frac{(d-1)^2}{b^2} + \frac{(a-d)(d-1)}{b} - c = 0,$$

and multiplying both sides by b gives

$$(d-1)^2 + (a-d)(d-1) = bc,$$

and hence

$$(a-1)(d-1) = bc.$$

Therefore, in both cases, we have $(a-1)(d-1) = bc$, as desired.