2019.3 Question 2

1. Let $y = 0$, and we have
$$f(x+0) = f(x) = f(x)f(0),$$
so either $f(x) = 0$ or $f(0) = 1$ for all x.
Assume, B.W.O.C., that $f(0) \neq 1$, then we must have $f(x) = 0$ for all x, which means $f'(x) = 0$, contradicting with $f'(0) = k \neq 0$.
Hence, $f(0) = 1$.
By definition of the derivative, we have
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h}, \end{aligned}$$
and letting $x = 0$, we also have
$$k = f'(0) = f(0) \lim_{h \to 0} \frac{f(h) - 1}{h} = \lim_{h \to 0} \frac{f(h) - 1}{h},$$
and hence
$$f'(x) = kf(x)$$
as desired.
This differential equation solves to
$$f(x) = Ae^{kx},$$
and with the condition $f(0) = 1$, we have $A = 1$, and hence
$$f(x) = e^{kx}$$
for all x.

2. Let $y = 0$, and we have
$$g(x+0) = g(x) = \frac{g(x) + g(0)}{1 + g(x)g(0)}.$$
This means that
$$g(x) + g(x)^2 g(0) = g(x) + g(0),$$
which gives
$$g(0) \left[g(x)^2 - 1 \right] = 0.$$
Since $|g(x)| < 1$ for all x, we must have $g(x)^2 - 1 < 0$, and hence $g(0) = 0$.
By the definition of the derivative,
$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{g(x)+g(h)}{1+g(x)g(h)} - g(x)}{h} \\ &= \lim_{h \to 0} \frac{g(x) + g(h) - g(x) - g(x)^2 g(h)}{h(1 + g(x)g(h))} \\ &= \lim_{h \to 0} \frac{g(h) \left[1 - g(x)^2 \right]}{h(1 + g(x)g(h))} \\ &= \left[1 - g(x)^2 \right] \lim_{h \to 0} \frac{g(h)}{h(1 + g(x)g(h))}. \end{aligned}$$

Considering the limit, we have

$$\begin{aligned}\lim_{h\to 0}\frac{g(h)}{h(1+g(x)g(h))} &= \lim_{h\to 0}\frac{g(h)/h}{1+g(x)g(h)}\\ &= \frac{\lim_{h\to 0}\left[g(h)/h\right]}{\lim_{h\to 0}\left[1+g(x)g(h)\right]}\\ &= \frac{\lim_{h\to 0}\left[g(h)/h\right]}{1}\\ &= \lim_{h\to 0}\frac{g(h)}{h},\end{aligned}$$

and hence

$$g'(x) = \left[1-g(x)^2\right]\lim_{h\to 0}\frac{g(h)}{h}.$$

Let $x=0$, and we have

$$k = g'(0) = 1\cdot\lim_{h\to 0}\frac{g(h)}{h},$$

hence giving the differential equation

$$g'(x) = k\left[1-g(x)^2\right].$$

This rearranges to give

$$\frac{\mathrm{d}g(x)}{1-g(x)^2} = k\,\mathrm{d}x,$$

and hence

$$\left[\frac{1}{1+g(x)}+\frac{1}{1-g(x)}\right]\mathrm{d}g(x) = 2k\,\mathrm{d}x,$$

which gives

$$\ln|1+g(x)| - \ln|1-g(x)| = 2kx + C.$$

Let $x=0$, we have $g(0)=0$, and hence $C=0$, and hence

$$\frac{1+g(x)}{1-g(x)} = \exp(2kx),$$

and hence

$$1+g(x) = \exp(2kx) - \exp(2kx)g(x),$$

which gives

$$g(x) = \frac{\exp(2kx)-1}{\exp(2kx)+1} = \frac{\exp(kx)-\exp(-kx)}{\exp(kx)+\exp(-kx)} = \tanh(kx).$$