2019.3 Question 12

For each integer between 1 to n inclusive, they are either in a subset of S, an element of T, or not. For each integer there are 2 choices, and there are n integers, this means that

$$|T| = 2^n,$$

as desired.

1. Since there is an equal number of sets $B \in T$ for $1 \in B$ and $1 \notin B$, this means

$$\mathrm{P}(1 \in A_1) = \frac{1}{2}.$$

2. For each of the integer $1 \leq t \leq n$, $t \notin A_1 \cap A_2$ if and only if they cannot be in both of A_1 and A_2, and hence

$$\mathrm{P}(t \notin A_1 \cap A_2) = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4},$$

and $A_1 \cap A_2 = \varnothing$ if and only if for all $1 \leq t \leq n$, that $t \notin A_1 \cap A_2$. All these events are independent, and hence

$$\mathrm{P}(A_1 \cap A_2 = \varnothing) = \left(\frac{3}{4}\right)^n.$$

By similar reasoning,

$$\mathrm{P}(A_1 \cap A_2 \cap A_3 = \varnothing) = \left(\frac{7}{8}\right)^n,$$

and

$$\mathrm{P}(A_1 \cap A_2 \cap \cdots \cap A_m = \varnothing) = \left[1 - \left(\frac{1}{2}\right)^m\right]^n = \left(1 - \frac{1}{2^m}\right)^n.$$

3. $A_1 \subseteq A_2$ if and only if for any $1 \leq t \leq n$, we have $t \in A_1 \implies t \in A_2$. For this to happen, either $t \notin A_1$ (in which case we do not worry about whether t is in A_2 or not), or $t \in A_1$ and $t \in A_2$. This means

$$\mathrm{P}(t \in A_1 \implies t \in A_2) = \frac{3}{4},$$

and hence

$$\mathrm{P}(A_1 \subseteq A_2) = \left(\frac{3}{4}\right)^n.$$

For any $1 \leq t \leq n$, $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m$ means we have $t \in A_1 \implies t \in A_2 \implies \cdots \implies t \in A_m$. This happens if and only if $t \in A_i$ gives $t \in A_j$ for all $j \geq i$, and this is true if and only if there exists some $0 \leq k \leq m$, such that for $1 \leq i \leq k$, $t \notin A_k$, and for $k < j \leq m$, $t \in A_k$.

There are precisely $m + 1$ choices for such k, and this means

$$\mathrm{P}(t \in A_1 \implies t \in A_2 \implies \cdots \implies t \in A_m) = \frac{m+1}{2^m},$$

and hence

$$\mathrm{P}(A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m) = \left(\frac{m+1}{2^m}\right)^n,$$

which gives

$$\mathrm{P}(A_1 \subseteq A_2 \subseteq A_3) = \left(\frac{1}{2}\right)^n.$$