2019.3 Question 11

1. Let X be the number of customers arriving at builders' merchants on a day, and we have $X \sim \mathrm{Po}(\lambda)$. This means
$$\mathrm{P}(X = x) = \frac{\lambda^x}{e^\lambda x!}$$
for $x = 0, 1, \ldots$.

Let Y be the number of customers taking the sand on a day. Then we have $(Y \mid X = x) \sim \mathrm{B}(x, p)$, and hence
$$\mathrm{P}(Y = y \mid X = x) = \binom{x}{y} p^y (1-p)^{x-y}.$$

Hence, we have
$$\begin{aligned}
\mathrm{P}(Y = y) &= \sum_{x=0}^{\infty} \mathrm{P}(Y = y, X = x) \\
&= \sum_{x=0}^{\infty} \mathrm{P}(Y = y \mid X = x)\,\mathrm{P}(X = x) \\
&= \sum_{x=y}^{\infty} \mathrm{P}(Y = y \mid X = x)\,\mathrm{P}(X = x) \\
&= \sum_{x=y}^{\infty} \binom{x}{y} p^y (1-p)^{x-y} \cdot \frac{\lambda^x}{e^\lambda x!} \\
&= \sum_{x=y}^{\infty} \frac{x! p^y (1-p)^x \lambda^x}{y!(x-y)!(1-p)^y e^\lambda x!} \\
&= \frac{p^y}{y!(1-p)^y e^\lambda} \sum_{x=y}^{\infty} \frac{(1-p)^x \lambda^x}{(x-y)!} \\
&= \frac{p^y}{y!(1-p)^y e^\lambda} \sum_{x=0}^{\infty} \frac{[\lambda(1-p)]^{x+y}}{x!} \\
&= \frac{p^y \lambda^y}{y! e^\lambda} \sum_{x=0}^{\infty} \frac{[\lambda(1-p)]^{x}}{x!} \\
&= \frac{(p\lambda)^y}{y! e^\lambda} e^{\lambda(1-p)} \\
&= \frac{(p\lambda)^y}{y! e^{p\lambda}},
\end{aligned}$$
which is precisely the probability mass function of $\mathrm{Po}(p\lambda)$, as desired.

2. Let Z be the amount of sand remaining at the end of a day, and hence
$$Z = S(1-k)^Y.$$

Hence, the expectation of Z is given by
$$\begin{aligned}
\mathrm{E}(Z) &= S\,\mathrm{E}\left[(1-k)^Y\right] \\
&= S \sum_{y=0}^{\infty} (1-k)^y\,\mathrm{P}(Y = y) \\
&= \frac{S}{e^{p\lambda}} \sum_{y=0}^{\infty} \frac{(p\lambda(1-k))^y}{y!} \\
&= \frac{S}{e^{p\lambda}} e^{p\lambda(1-k)} \\
&= \frac{S}{e^{pk\lambda}}.
\end{aligned}$$

Let Z' be the amount of sand taken, and hence

$$Z' = S - Z,$$

which means

$$\mathrm{E}(Z') = S - \mathrm{E}(Z) = S\left(1 - e^{-pk\lambda}\right),$$

precisely as desired.

3. Given that $Z = z$, the assistant will take kz of the remaining sand, and the probability of the assistant taking the golden grain event (denoted as G) is

$$\mathrm{P}(G \mid Z = z) = \frac{kz}{S}.$$

Using $Z = S(1-k)^Y$, we have

$$\mathrm{P}(G \mid Y = y) = k(1-k)^y$$

$$\begin{aligned}
\mathrm{P}(G) &= \sum_{y=0}^{\infty} \mathrm{P}(G, Y = y) \\
&= \sum_{y=0}^{\infty} \mathrm{P}(G \mid Y = y)\,\mathrm{P}(Y = y) \\
&= \sum_{y=0}^{\infty} k(1-k)^y \cdot \frac{(p\lambda)^y}{y! e^{p\lambda}} \\
&= \frac{k}{e^{p\lambda}} \sum_{y=0}^{\infty} \frac{(p\lambda(1-k))^y}{y!} \\
&= \frac{k}{e^{p\lambda}} e^{p\lambda(1-k)} \\
&= \frac{k}{e^{pk\lambda}}.
\end{aligned}$$

In the case where $k = 0$, no sand is taken, and hence the probability is 0.

In the case where $k \to 1$, $\mathrm{P}(G) = e^{-p\lambda}$, which is the probability that $Y = 0$. This is precisely when no customer takes any sand (since if any took the sand they must have taken the gold grain), and as $k \to 1$ the merchants' assistant is guaranteed to take the gold provided it is still existent in the final pile.

In the case where $p\lambda > 1$, we differentiate the probability with respect to k, which gives

$$\frac{\mathrm{d}ke^{-pk\lambda}}{\mathrm{d}k} = (1 - pk\lambda)e^{-pk\lambda}.$$

$e^{-pk\lambda}$ is always positive. In the case where $k < \frac{1}{p\lambda}$, $1 - pk\lambda > 0$, and when $k > \frac{1}{p\lambda}$, $1 - pk\lambda < 0$. Hence, precisely when $k = \frac{1}{p\lambda}$, we will have $\mathrm{P}(G)$ taking a maximum, and since $p\lambda > 1$, this k will satisfy $0 < k < 1$ which is within the range.

Hence, the value of k that maximises $\mathrm{P}(G)$ is

$$k = \frac{1}{p\lambda}.$$