2019.3 Question 1

1. When $k = 1$,

$$\dot{x} = -x - y, \dot{y} = x - y.$$

Hence,

$$\begin{aligned}\ddot{x} &= -\dot{x} - \dot{y} \\ &= -\dot{x} - (x - y) \\ &= -\dot{x} - x + y \\ &= -\dot{x} - x + (-x - \dot{x}) \\ &= -2\dot{x} - 2x,\end{aligned}$$

and this gives

$$\ddot{x} + 2\dot{x} + 2x = 0.$$

The auxiliary equation to this differential equation is

$$\lambda^2 + 2\lambda + 2 = 0,$$

which solves to

$$\lambda = -1 \pm i.$$

The general solution for x is hence

$$x(t) = \exp(-t)\left(A \sin t + B \cos t\right).$$

This means

$$\begin{aligned}\dot{x}(t) &= -\exp(-t)\left(A \sin t + B \cos t\right) + \exp(-t)\left(A \cos t - B \sin t\right) \\ &= -x(t) + \exp(-t)\left(A \cos t - B \sin t\right),\end{aligned}$$

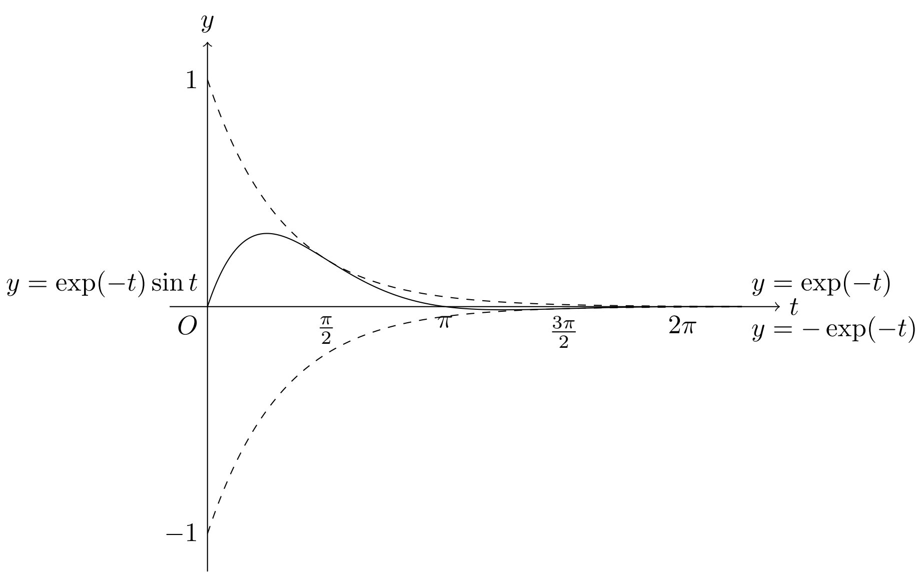
and hence

$$y(t) = -\exp(-t)\left(A \cos t - B \sin t\right) = \exp(-t)\left(B \sin t - A \cos t\right).$$

When $t = 0$, $x = x(0) = B = 1$, $y = y(0) = -A = 0$. Hence,

$$x(t) = \exp(-t) \cos t, y(t) = \exp(-t) \sin t.$$

The graph of y against t looks as follows:



y is greatest at the first stationary point of y, as shown in the graph. Note that

$$\dot{y} = x - y = \exp(-t)\left(\cos t - \sin t\right),$$

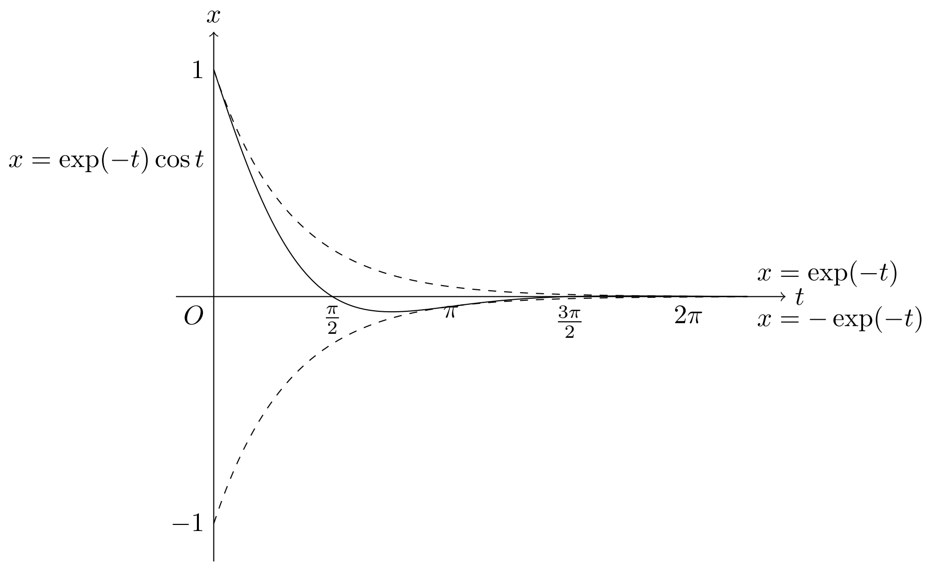
and hence

$$\dot{y} = 0 \iff \cos t = \sin t \iff \tan t = 1,$$

and the smallest positive solution to this is $t = \frac{\pi}{4}$. The coordinate of the point is hence

$$(x, y) = \left(\exp\left(-\frac{\pi}{4}\right) \cdot \frac{\sqrt{2}}{2}, \exp\left(-\frac{\pi}{4}\right) \cdot \frac{\sqrt{2}}{2}\right).$$

Similarly, the graph of x against t looks as follows:



x is smallest at the first stationary point of x, as shown in the graph. Note that

$$\dot{x} = -x - y = -\exp(-t)\left(\cos t + \sin t\right),$$

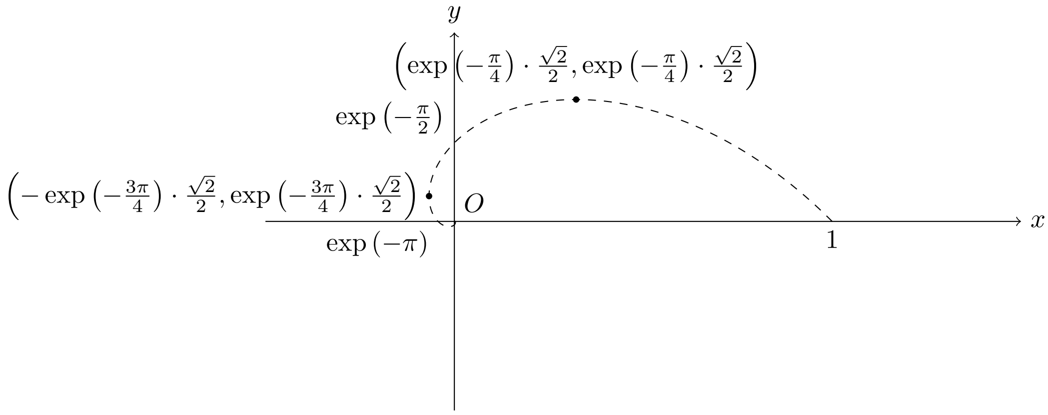
and hence

$$\dot{x} = 0 \iff \cos t = -\sin t \iff \tan t = -1,$$

and the smallest positive solution to this is $t = \frac{3\pi}{4}$. The coordinate of the point is hence

$$(x, y) = \left(-\exp\left(-\frac{3\pi}{4}\right) \cdot \frac{\sqrt{2}}{2}, \exp\left(-\frac{3\pi}{4}\right) \cdot \frac{\sqrt{2}}{2}\right).$$

Without the $\exp(-t)$ factor, the x-y graph will simply be a circle, and with this factor, it will be a spiral with exponentially decreasing radius. This is the polar curve $r = \exp(-\theta)$. Hence, the x-y graph looks as follows.



2. Since $\dot{x} = -x$, we must have $x(t) = A\exp(-t)$, and since $x(0) = 1$, we have $A = 1$ and $x(t) = \exp(-t)$.

   We have
$$\dot{y} = \exp(-t) - y,$$
   and hence
$$\dot{y} + y = \exp(-t).$$
   Multiplying both sides by $\exp(t)$, we have
$$e^t\dot{y} + e^t y = 1,$$
   and hence
$$\frac{\mathrm{d}ye^t}{\mathrm{d}t} = 1,$$
   which gives
$$ye^t = t + B,$$
   and hence
$$y = \exp(-t)(t + B).$$
   Since $y = 0$ when $t = 0$, we must have $B = 0$, and hence
$$y = t\exp(-t).$$
   Note that
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \exp(-t) - t\exp(-t),$$
   and hence $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ when $t = 1$, which is when
$$(x, y) = \left(e^{-1}, e^{-1}\right).$$
   Note that
$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\exp(-t),$$
   and hence $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$, which is when
$$(x, y) = (1, 0),$$
   and the tangent to the curve at this point will be vertical.

   Hence, the graph will look as follows: