## 2018.3 Question 8

1. Using the substitution $t = \frac{1}{x}$, we have

$$\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{1}{x^2} \implies \mathrm{d}x = -x^2\,\mathrm{d}t = -\frac{\mathrm{d}t}{t^2},$$

and when $x \to 0^+$, $t \to \infty$, and when $x = 1$, $t = 1$. Hence,

$$\begin{aligned}
I &= \int_0^1 \frac{f(x^{-1})}{1+x}\,\mathrm{d}x \\
&= \int_1^\infty \frac{f(t)}{1+t^{-1}} \cdot \left(-\frac{\mathrm{d}t}{t^2}\right) \\
&= \int_1^\infty \frac{f(t)\,\mathrm{d}t}{t(1+t)} \\
&= \int_1^2 \frac{f(t)\,\mathrm{d}t}{t(1+t)} + \int_2^3 \frac{f(t)\,\mathrm{d}t}{t(1+t)} + \int_3^4 \frac{f(t)\,\mathrm{d}t}{t(1+t)} + \cdots \\
&= \sum_{n=1}^\infty \int_n^{n+1} \frac{f(t)\,\mathrm{d}t}{t(1+t)},
\end{aligned}$$

as desired.

Since $f(x) = f(x+1)$ for all $x$, we must have that $f(x) = f(x+n)$ for all $x$ and integers $n$. Also, we have

$$\frac{1}{y(1+y)} = \frac{1}{y} - \frac{1}{1+y}.$$

Hence,

$$\begin{aligned}
I &= \sum_{n=1}^\infty \int_n^{n+1} \frac{f(t)\,\mathrm{d}t}{t(1+t)} \\
&= \sum_{n=1}^\infty \int_0^1 \frac{f(n+t)\,\mathrm{d}t}{(n+t)(n+t+1)} \\
&= \sum_{n=1}^\infty \int_0^1 f(t) \cdot \left[\frac{1}{n+t} - \frac{1}{n+t+1}\right]\mathrm{d}t \\
&= \sum_{n=1}^\infty \int_0^1 \frac{f(t)\,\mathrm{d}t}{n+t} - \sum_{n=1}^\infty \int_0^1 \frac{f(t)\,\mathrm{d}t}{n+t+1} \\
&= \sum_{n=1}^\infty \int_0^1 \frac{f(t)\,\mathrm{d}t}{n+t} - \sum_{n=2}^\infty \int_0^1 \frac{f(t)\,\mathrm{d}t}{n+t} \\
&= \int_0^1 \frac{f(t)\,\mathrm{d}t}{1+t}.
\end{aligned}$$

2. For the first integral, simply consider $f(x) = \{x\}$, and we can immediately see that $f(x)$ has period of 1 from the definition. Hence,

$$\int_0^1 \frac{\{x^{-1}\}}{1+x}\,\mathrm{d}x = \int_0^1 \frac{f(x^{-1})}{1+x}\,\mathrm{d}x = \int_0^1 \frac{f(x)}{1+x}\,\mathrm{d}x = \int_0^1 \frac{\{x\}}{1+x}\,\mathrm{d}x.$$

Since for $0 < x < 1$, we have $\{x\} = x$, and hence

$$\begin{aligned}\int_0^1 \frac{\{x^{-1}\}}{1+x}\,\mathrm{d}x &= \int_0^1 \frac{\{x\}}{1+x}\,\mathrm{d}x \\ &= \int_0^1 \frac{x}{1+x}\,\mathrm{d}x \\ &= \int_0^1 \left(1 - \frac{1}{1+x}\right)\mathrm{d}x \\ &= 1 - [\ln(1+x)]_0^1 \\ &= 1 - (\ln(2) - \ln(1)) \\ &= 1 - \ln 2.\end{aligned}$$

For the second integral, we let $g(x) = \{2x\}$, and we can see that $g(x)$ has a period of $\frac{1}{2}$, and hence it also has a period of 1. Hence,

$$\int_0^1 \frac{\{2x^{-1}\}}{1+x}\,\mathrm{d}x = \int_0^1 \frac{g(x^{-1})}{1+x}\,\mathrm{d}x = \int_0^1 \frac{g(x)}{1+x}\,\mathrm{d}x = \int_0^1 \frac{\{2x\}}{1+x}\,\mathrm{d}x.$$

We split this integral into two parts, $[0, 0.5]$ and $[0.5, 1]$.

$$\begin{aligned}\int_0^1 \frac{\{2x^{-1}\}}{1+x}\,\mathrm{d}x &= \int_0^1 \frac{\{2x\}}{1+x}\,\mathrm{d}x \\ &= \int_0^{0.5} \frac{\{2x\}}{1+x}\,\mathrm{d}x + \int_{0.5}^1 \frac{\{2x\}}{1+x}\,\mathrm{d}x \\ &= \int_0^{0.5} \frac{2x}{1+x}\,\mathrm{d}x + \int_{0.5}^1 \frac{2x-1}{1+x}\,\mathrm{d}x \\ &= \int_0^{0.5} \left[2 - \frac{2}{1+x}\right]\mathrm{d}x + \int_{0.5}^1 \left[2 - \frac{3}{1+x}\right]\mathrm{d}x \\ &= 1 - 2\left[\ln(1+x)\right]_0^{0.5} + 1 - 3\left[\ln(1+x)\right]_{0.5}^1 \\ &= 2 - 2\ln 1.5 + 2\ln 1 - 3\ln 2 + 3\ln 1.5 \\ &= 2 - 3\ln 2 + \ln 1.5 \\ &= 2 - 3\ln 2 + \ln 3 - \ln 2 \\ &= 2 - 4\ln 2 + \ln 3.\end{aligned}$$