2018.3 Question 7

1. We have

$$\begin{aligned}
&\frac{(\cot\theta+i)^{2n+1}-(\cot\theta-i)^{2n+1}}{2i}\\
&=\frac{(\cos\theta+i\sin\theta)^{2n+1}-(\cos\theta-i\sin\theta)^{2n+1}}{2i\sin^{2n+1}\theta}\\
&=\frac{(\cos(2n+1)\theta+i\sin(2n+1)\theta)-(\cos(2n+1)\theta-i\sin(2n+1)\theta)}{2i\sin^{2n+1}\theta}\\
&=\frac{2i\sin(2n+1)\theta}{2i\sin^{2n+1}\theta}\\
&=\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta},
\end{aligned}$$

as desired.

By applying the binomial expansion formula on the numerator, we have

$$\begin{aligned}
&(\cot\theta+i)^{2n+1}-(\cot\theta-i)^{2n+1}\\
&=\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot i^{2n+1-t}-\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot(-i)^{2n+1-t}\\
&=\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot\left[i^{2n+1-t}-(-i)^{2n+1-t}\right]\\
&=(-1)^n\cdot i\cdot\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot i^{-t}\cdot\left[1-(-1)^{1-t}\right].
\end{aligned}$$

Due to the existence of the final term, this means that only terms with even t will retain (give a 2), and odd ts will cancel. Hence,

$$\begin{aligned}
&(\cot\theta+i)^{2n+1}-(\cot\theta-i)^{2n+1}\\
&=(-1)^n\cdot i\cdot\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot i^{-t}\cdot\left[1-(-1)^{1-t}\right]\\
&=(-1)^n\cdot 2i\cdot\sum_{t=0}^{n}\binom{2n+1}{2t}\cot^{2t}\theta\cdot i^{-2t}\\
&=2i(-1)^n\cdot\sum_{t=0}^{n}\binom{2n+1}{2t}\cot^{2t}\theta\cdot(-1)^t\\
&=2i(-1)^n\cdot\sum_{t=0}^{n}\binom{2n+1}{2n-2t+1}\cot^{2t}\theta\cdot(-1)^t\\
&=2i(-1)^n\cdot\sum_{t=0}^{n}\binom{2n+1}{2t+1}\cot^{2(n-t)}\theta\cdot(-1)^{n-t}\\
&=2i\cdot\sum_{t=0}^{n}\binom{2n+1}{2t+1}\cot^{2(n-t)}\theta\cdot(-1)^t.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta}\\
&=\frac{2i\cdot\sum_{t=0}^{n}\binom{2n+1}{2t+1}\cot^{2(n-t)}\theta\cdot(-1)^t}{2i}\\
&=\sum_{t=0}^{n}\binom{2n+1}{2t+1}\cot^{2(n-t)}\theta\cdot(-1)^t.
\end{aligned}$$

The left-hand side of the original equation is

$$\sum_{t=0}^{n} \binom{2n+1}{2t+1} x^{n-t} \cdot (-1)^t.$$

Let $x = \cot^2 \theta$, we have

$$\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta} = \sum_{t=0}^{n} \binom{2n+1}{2t+1} x^{n-t} \cdot (-1)^t = 0.$$

Therefore, we have $\sin(2n+1)\theta = 0$, and hence $(2n+1)\theta = m\pi$ for $m \in \mathbb{Z}$.

To avoid duplicate solutions for $x = \cot^2 \theta$, we restrict $\theta \in \left(0, \frac{\pi}{2}\right]$, and hence $(2n+1)\theta \in \left(0, \left(n + \frac{1}{2}\right)\pi\right]$, and hence $m = 1, 2, \ldots, n$.

This solves to $\theta = \frac{m\pi}{2n+1}$ for $m = 1, 2, \ldots, n$, and hence this gives exactly

$$x = \cot^2 \left(\frac{m\pi}{2n+1}\right).$$

2. By Vieta's Theorem, we will have

$$\sum_{m=1}^{n} x_m = -\frac{-\binom{2n+1}{3}}{\binom{2n+1}{1}} = \frac{(2n+1)(2n)(2n-1)}{(2n+1)\cdot 3 \cdot 2 \cdot 1} = \frac{n(2n-1)}{3},$$

and since we have

$$x_m = \cot^2\left(\frac{m\pi}{2n+1}\right),$$

we have

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) = \frac{n(2n-1)}{3}.$$

3. For $0 < \theta < \frac{1}{2}\pi$, we have $0 < \sin\theta < \theta < \tan\theta$, and squaring this gives

$$0 < \sin^2\theta < \theta^2 < \tan^2\theta,$$

and flipping to the reciprocal gives

$$0 < \cot^2\theta < \frac{1}{\theta^2} < \csc^2\theta = 1 + \cot^2\theta,$$

which proves exactly what is desired.

Therefore, we have

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) < \sum_{m=1}^{n} \frac{1}{\left(\frac{m\pi}{2n+1}\right)^2} < \sum_{m=1}^{n} \left[1 + \cot^2\left(\frac{m\pi}{2n+1}\right)\right],$$

and hence

$$\frac{n(2n-1)}{3} < \sum_{m=1}^{n} \frac{(2n+1)^2}{m^2\pi^2} < \frac{2n(n+1)}{3},$$

and hence

$$\frac{n(2n-1)\pi^2}{3(2n+1)^2} < \sum_{m=1}^{n} \frac{1}{m^2} < \frac{2n(n+1)\pi^2}{3(2n+1)^2}.$$

Take the limit as $n \to \infty$, the strict inequalities become weak, and hence

$$\lim_{n\to\infty} \frac{n(2n-1)\pi^2}{3(2n+1)^2} \leq \sum_{m=1}^{\infty} \frac{1}{m^2} \leq \lim_{n\to\infty} \frac{2n(n+1)\pi^2}{3(2n+1)^2},$$

and hence

$$\frac{2\pi^2}{3 \cdot 2^2} \leq \sum_{m=1}^{\infty} \frac{1}{m^2} \leq \frac{2n\pi^2}{3 \cdot 2^2},$$

and therefore

$$\frac{\pi^2}{6} \leq \sum_{m=1}^{\infty} \frac{1}{m^2} \leq \frac{\pi^2}{6},$$

and hence

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6},$$

as desired.