2018.3 Question 6

1. Since A, Q, C lie on a straight line, $\mathbf{AQ} = \lambda \mathbf{AC}$ for some $\lambda \in \mathbb{R}$. This means

   $$q - a = \lambda(c - a),$$

   and hence

   $$\frac{q-a}{c-a} = \lambda \in \mathbb{R},$$

   as required.

   Hence, we must have

   $$\frac{q-a}{c-a} = \left(\frac{q-a}{c-a}\right)^* = \frac{q^*-a^*}{c^*-a^*}.$$

   Cross-multiplying the terms out give

   $$(c-a)(q^*-a^*) = (c^*-a^*)(q-a)$$

   exactly as desired.

   Substituting in $a^* = 1/a$ and $c^* = 1/c$, we have

   $$(c-a)\left(q^* - \frac{1}{a}\right) = \left(\frac{1}{c} - \frac{1}{a}\right)(q-a),$$

   and expanding the brackets gives

   $$cq^* - aq^* - \frac{c}{a} + 1 = \frac{q}{c} - \frac{a}{c} - \frac{q}{a} + 1,$$

   and hence

   $$cq^* - aq^* - \frac{c}{a} = \frac{q}{c} - \frac{a}{c} - \frac{q}{a}.$$

   Multiplying by ac on both sides gives us

   $$ac^2q^* - a^2cq^* - c^2 = aq - a^2 - cq,$$

   and hence

   $$ac(c-a)q^* = (a-c)q - (a^2-c^2) = (a-c)q - (a-c)(a+c).$$

   We can divide through $(a-c)$ on both sides since $a \neq c$. Hence,

   $$0 = q - (a+c) + acq^*,$$

   and hence

   $$acq^* + q = a + c,$$

   as desired.

2. By part 1, we must have

   $$acq^* + q = a + c, bdq^* + q = b + d.$$

   Since $q = q$, we have

   $$acq^* - (a+c) = bdq^* - (b+d),$$

   and rearranging gives

   $$(ac - bd)q^* = (a+c) - (b+d),$$

   exactly as desired.

   We also have $q^* = q^*$, and hence

   $$\frac{a+c-q}{ac} = \frac{b+d-q}{bd},$$

   which gives

   $$(bd)(a+c-q) = (ac)(b+d-q),$$

and rearranging gives

$$(ac - bd)q = ac(b + d) - bd(a + c).$$

Summing this with previously, we have

$$(ac - bd)(q + q^*) = (a + c) - (b + d) + ac(b + d) - bd(a + c).$$

We notice that

$$\begin{aligned}(a + c) - (b + d) + ac(b + d) - bd(a + c) &= a + c - b - d + abc + acd - abd - bcd \\ &= a - b + acd - bcd + c - d + abc - abd \\ &= (a - b)(1 + cd) + (c - d)(1 + ab),\end{aligned}$$

and hence

$$(ac - bd)(q + q^*) = (a - b)(1 + cd) + (c - d)(1 + ab),$$

exactly as desired.

3. By part 1, we must have

$$p + abp^* = a + b.$$

Since p is real, $p = p^*$, and hence

$$(1 + ab)p = a + b,$$

as desired.

Similarly, we must have

$$(1 + cd)q = c + d,$$

and putting this back into the result from part 2, we have

$$(ac - bd)(q + q^*) = \frac{(a - b)(c + d)}{p} + \frac{(c - d)(a + b)}{p},$$

and hence since $ac - bd \neq 0$, we have

$$\begin{aligned}p(q + q^*) &= \frac{(a - b)(c + d) + (c - d)(a + b)}{ac - bd} \\ &= \frac{ac + ad - bc - bd + ac + bc - ad - bd}{ac - bd} \\ &= \frac{2ac - 2bd}{ac - bd} \\ &= 2,\end{aligned}$$

as desired.