2018.3 Question 5

1. First, we notice that

$$G_{k+1}^{k+1} = \prod_{t=1}^{k+1} a_t = a_{k+1} G_k^k,$$

and hence

$$G_{k+1} = \left(a_{k+1} G_k^k\right)^{\frac{1}{k+1}}.$$

Similarly, notice that

$$(k+1) A_{k+1} = \sum_{t=1}^{k+1} a_t = a_{k+1} + kA_k.$$

Hence,

$$\begin{aligned}(k+1)\left(A_{k+1} - G_{k+1}\right) &\geq k\left(A_k - G_k\right),\\ a_{k+1} + kA_k - (k+1)\left(a_{k+1}G_k^k\right)^{\frac{1}{k+1}} &\geq ka_k - kG_k,\\ a_{k+1} + kG_k &\geq (k+1) a_{k+1}^{\frac{1}{k+1}} G_k^{\frac{k}{k+1}}.\end{aligned}$$

Dividing both sides by G_k, we have

$$\begin{aligned}\frac{a_{k+1}}{G_k} + k &\geq (k+1) a_{k+1}^{\frac{1}{k+1}} G_k^{-\frac{1}{k+1}},\\ \lambda_k^{k+1} + k &\geq (k+1)\left(\frac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}},\\ \lambda_k^{k+1} + k &\geq (k+1)\lambda_k,\\ \lambda_k^{k+1} - (k+1)\lambda_k + k &\geq 0,\end{aligned}$$

as desired. (Notice that the condition for the equal sign is equivalent as well.)

2. By differentiation, we have

$$f'(x) = (k+1)x^k - (k+1) = (k+1)(x^k - 1).$$

When $x \in (0,1), x^k \in (0,1), f'(x) < 0$, and hence f is strictly decreasing.
When $x \in (1,\infty), x^k \in (1,\infty), f'(x) > 0$, and hence f is strictly increasing.
Hence, $f(1)$ is the minimum for f on $(0,\infty)$. This means for all $x \in (0,\infty)$, we have

$$f(x) \geq f(1) = 1^{k+1} - (k+1) + k = 0,$$

taking the equal sign if and only if $x = 1$.

3. (a) We show this by induction. For the base case $n = 1$, $A_1 = G_1 = a_1$, so naturally $A_n \geq G_n$ is satisfied.
Assume that the statement holds for some $n = k$, i.e. $A_k \geq G_k$, $A_k - G_k \geq 0$. Since $k > 0$ as well, we must have

$$(k+1)(A_{k+1} - G_{k+1}) \geq k(A_k - G_k) \geq 0.$$

We also have $k + 1 > 0$, and hence

$$A_{k+1} - G_{k+1} \geq 0 \iff A_{k+1} \geq G_{k+1},$$

meaning the statement holds for $n = k + 1$ as well.
Hence, by the principle of mathematical induction, we must have $A_n \geq G_n$ for all $n \in \mathbb{N}$, which finishes our proof.

(b) We show this by induction. For the base case $n = 1$, this condition is naturally satisfied.
Assume that the statement holds for some $n = k$, i.e. $A_k = G_k \implies a_1 = a_2 = \cdots = a_k$. We show this for $n = k + 1$. If $A_{k+1} = G_{k+1}$, then we must have

$$k(A_k - G_k) \leq (k+1)(A_{k+1} - G_{k+1}) = 0,$$

but since $A_k \geq G_k$, we must have then $A_k = G_k$, and hence the equal sign in the inequality being taken.
This must mean that

$$\lambda_k = \left(\frac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}} = 1,$$

and hence

$$a_{k+1} = G_k.$$

At the same time, since $A_k = G_k$, we must have $a_1 = a_2 = \cdots = a_k$, and hence $G_k = a_1 = a_2 = \cdots = a_k$. Therefore, we must also have

$$a_1 = a_2 = \cdots = a_k = a_{k+1},$$

which proves the statement that $A_{k+1} = G_{k+1}$ implies $a_1 = a_2 = \cdots = a_k = a_{k+1}$, which is the original statement for $n = k + 1$.
Hence, by the principle of mathematical induction, we must have $A_n = G_n$ implies $a_1 = a_2 = \cdots = a_n$ for all $n \in \mathbb{N}$, which finishes our proof.