2018.3 Question 4

The hyperbola has parametric equation

$$\begin{cases} x = a\sec\theta, \\ y = b\tan\theta. \end{cases}$$

Hence, by differentiation, we have

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} \\ &= \frac{b\sec^2\theta}{a\sec\theta\tan\theta} \\ &= \frac{b\cos\theta}{a\sin\theta\cos\theta} \\ &= \frac{b}{a\sin\theta}. \end{aligned}$$

The tangent to the hyperbola at P will be

$$y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta),$$

which simplifies to

$$ay\sin\theta - ab\tan\theta\sin\theta = bx - ab\sec\theta,$$

and hence

$$bx - ay\sin\theta = ab(\sec\theta - \tan\theta\sin\theta).$$

Notice that

$$\sec\theta - \tan\theta\sin\theta = \frac{1-\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta}{\cos\theta} = \cos\theta,$$

and so the equation of the tangent is

$$bx - ay\sin\theta = ab\cos\theta,$$

exactly as desired.

1. Let $\frac{x}{a} = \frac{y}{b} = s$ for S, we have $x = as$ and $y = bs$, and hence

   $$abs - abs\sin\theta = ab\cos\theta,$$

   which gives

   $$s = \frac{\cos\theta}{1-\sin\theta},$$

   and hence

   $$S\left(a\frac{\cos\theta}{1-\sin\theta}, b\frac{\cos\theta}{1-\sin\theta}\right).$$

   Let $\frac{x}{a} = -\frac{y}{b} = t$ for T, we have $x = at$ and $y = -bt$, and hence

   $$abt + abt\sin\theta = ab\cos\theta,$$

   which gives

   $$t = \frac{\cos\theta}{1+\sin\theta},$$

   and hence

   $$T\left(a\frac{\cos\theta}{1+\sin\theta}, -b\frac{\cos\theta}{1+\sin\theta}\right).$$

We have

$$\begin{aligned}\frac{a\frac{\cos\theta}{1-\sin\theta}+a\frac{\cos\theta}{1+\sin\theta}}{2}&=\frac{a\cos\theta}{2}\left(\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}\right)\\&=\frac{a\cos\theta}{2}\left(\frac{2}{\cos^2\theta}\right)\\&=\frac{a}{\cos\theta}\\&=a\sec\theta,\end{aligned}$$

and

$$\begin{aligned}\frac{a\frac{\cos\theta}{1-\sin\theta}-b\frac{\cos\theta}{1+\sin\theta}}{2}&=\frac{b\cos\theta}{2}\left(\frac{1}{1-\sin\theta}-\frac{1}{1+\sin\theta}\right)\\&=\frac{b\cos\theta}{2}\left(\frac{2\sin\theta}{\cos^2\theta}\right)\\&=\frac{b\sin\theta}{\cos\theta}\\&=b\tan\theta.\end{aligned}$$

This means the midpoint of ST is $(a\sec\theta, b\tan\theta)$, which is exactly P.

2. Since the tangents are perpendicular, that means

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_\theta\cdot\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_\varphi=-1,$$

and hence

$$\frac{b}{a\sin\theta}\cdot\frac{b}{a\sin\varphi}=-1,$$

which means

$$b^2=-a^2\sin\theta\sin\varphi.$$

The two tangents are

$$bx-ay\sin\theta=ab\cos\theta$$

and

$$bx-ay\sin\varphi=ab\cos\varphi.$$

Since $bx=bx$, we have

$$ay\sin\theta+ab\cos\theta=ay\sin\varphi+ab\cos\varphi,$$

and hence

$$y(\sin\theta-\sin\varphi)=b(\cos\varphi-\cos\theta),$$

which gives

$$y=b\cdot\frac{\cos\varphi-\cos\theta}{\sin\theta-\sin\varphi}.$$

Hence,

$$\begin{aligned}x&=\frac{ab\cos\theta+ay\sin\theta}{b}\\&=\frac{a}{b}\left(b\cos\theta+b\sin\theta\frac{\cos\varphi-\cos\theta}{\sin\theta-\sin\varphi}\right)\\&=a\left(\cos\theta+\sin\theta\frac{\cos\varphi-\cos\theta}{\sin\theta-\sin\varphi}\right)\\&=a\frac{\cos\theta(\sin\theta-\sin\varphi)+\sin\theta(\cos\varphi-\cos\theta)}{\sin\theta-\sin\varphi}\\&=a\cdot\frac{\sin\theta\cos\varphi-\cos\theta\sin\varphi}{\sin\theta-\sin\varphi}\\&=a\cdot\frac{\sin(\theta-\varphi)}{\sin\theta-\sin\varphi}.\end{aligned}$$

This means

$$\begin{cases} x^2 = a^2 \cdot \dfrac{\sin^2(\theta - \varphi)}{(\sin\theta - \sin\varphi)^2}, \\ y^2 = b^2 \cdot \dfrac{(\cos\varphi - \cos\theta)^2}{(\sin\theta - \sin\varphi)^2} = -a^2 \sin\theta \sin\varphi \cdot \dfrac{(\cos\varphi - \cos\theta)^2}{(\sin\theta - \sin\varphi)^2}. \end{cases}$$

Notice that

$$a^2 - b^2 = a^2 + a^2 \sin\theta \sin\varphi = a^2(1 + \sin\theta \sin\varphi).$$

Hence,

$$\begin{aligned} x^2 + y^2 &= a^2 \left[\frac{\sin^2(\theta - \varphi)}{(\sin\theta - \sin\varphi)^2} - \sin\theta \sin\varphi \cdot \frac{(\cos\varphi - \cos\theta)^2}{(\sin\theta - \sin\varphi)^2} \right] \\ &= \frac{a^2}{(\sin\theta - \sin\varphi)^2} \left[\sin^2(\theta - \varphi) - \sin\theta \sin\varphi \, (\cos\varphi - \cos\theta)^2 \right]. \end{aligned}$$

What is desired is to show

$$(1 + \sin\theta \sin\varphi)(\sin\theta - \sin\varphi)^2 = \sin^2(\theta - \varphi) - \sin\theta \sin\varphi \, (\cos\varphi - \cos\theta)^2.$$

We have

$$\begin{aligned} \text{RHS} &= (\sin\theta \cos\varphi - \cos\theta \sin\varphi)^2 - \sin\theta \sin\varphi(\cos^2\varphi + \cos^2\theta - 2\cos\varphi \cos\theta) \\ &= \sin^2\theta \cos^2\varphi + \cos^2\theta \sin^2\varphi - 2\sin\theta \cos\theta \sin\varphi \cos\varphi \\ &\quad - \sin\theta \sin\varphi \cos^2\varphi - \sin\theta \sin\varphi \cos^2\theta + 2\sin\theta \cos\theta \sin\varphi \cos\varphi \\ &= \sin\theta \cos^2\varphi(\sin\theta - \sin\varphi) + \cos^2\theta \sin\varphi(\sin\varphi - \sin\theta) \\ &= (\sin\theta \cos^2\varphi - \cos^2\theta \sin\varphi)(\sin\theta - \sin\varphi). \end{aligned}$$

Therefore, what is left to prove is that

$$(1 + \sin\theta \sin\varphi)(\sin\theta - \sin\varphi) = \sin\theta \cos^2\varphi - \cos^2\theta \sin\varphi$$

Notice that

$$\begin{aligned} \text{LHS} &= \sin\theta - \sin\varphi + \sin^2\theta \sin\varphi - \sin\theta \sin^2\varphi \\ &= \sin\theta(1 - \sin^2\varphi) - \sin\varphi(1 - \sin^2\theta) \\ &= \sin\theta \cos^2\varphi - \sin\varphi \cos^2\theta \\ &= \text{RHS}. \end{aligned}$$

This shows that

$$\frac{1}{(\sin\theta - \sin\varphi)^2} \left[\sin^2(\theta - \varphi) - \sin\theta \sin\varphi \, (\cos\varphi - \cos\theta)^2 \right] = 1 + \sin\theta \sin\varphi,$$

and hence

$$x^2 + y^2 = a^2 - b^2,$$

as desired.