## 2018.3 Question 3

Notice that

$$\begin{aligned} x^a(x^b(x^cy)')' &= x^a(x^b(cx^{c-1}y+x^cy'))' \\ &= x^a\left[x^{b+c-1}\left(cy+xy'\right)\right]' \\ &= x^a\left[(b+c-1)x^{b+c-2}\left(cy+xy'\right)+x^{b+c-1}\left(cy'+y'+xy''\right)\right] \\ &= x^{a+b+c-2}\left[(b+c-1)\left(cy+xy'\right)+x\left(cy'+y'+xy''\right)\right] \\ &= x^{a+b+c-2}\left[x^2y''+(b+2c)xy'+(b+c-1)cy\right]. \end{aligned}$$

Comparing this with the left-hand side of the original equation, we must have

$$\begin{cases} a+b+c-2=0, \\ b+2c=1-2p, \\ (b+c-1)c=p^2-q^2. \end{cases}$$

The second equation gives

$$b=1-2p-2c,$$

and putting this into the third equation gives

$$(1-2p-2c+c-1)c=p^2-q^2,$$

and hence

$$c^2+2pc+p^2-q^2=0.$$

This gives

$$(c+(p-q))(c+(p+q))=0,$$

and hence

$$c_1=-p+q, c_2=-p-q.$$

Putting this back, we get

$$b_1=1-2p-2(-p+q)=1-2q, b_2=1-2p-2(-p-q)=1+2q,$$

and since $a=2-b-c$ from the first equation, we have

$$a_1=2-(1-2q)-(-p+q)=1+p+q$$

and

$$a_2=2-(1+2q)-(-p-q)=1+p-q$$

Hence, the solutions are

$$\begin{cases} a=p\pm q+1, \\ b=\mp 2q+1, \\ c=-p\pm q. \end{cases}$$

1. In the case where $f(x)=0$. We must have

   $$x^a\left(x^b(x^cy)'\right)'=0,$$

   and hence

   $$\left(x^b(x^cy)'\right)'=0.$$

   Therefore, we must have by integration

   $$x^b(x^cy)'=C_1$$

   for some (real) constant $C_1$.
   Hence,

   $$(x^cy)'=C_1x^{-b}.$$

   There are two cases here:

(a) When $b = 1$ i.e. $q = 0$, the right-hand side is $C_1 x^{-1}$, and the left-hand side is $(x^c y)'$. Integrating both sides give

$$x^c y = C_1 \ln x + C_2$$

for some (real) constant $C_2$.
Hence,

$$y = x^{-c}(C_1 \ln x + C_2)$$

for some (real) constants $C_1, C_2$.
When $q = 0$, $c = -p$, and hence

$$y = x^p(C_1 \ln x + C_2).$$

(b) When $b \neq 1$ i.e. $q \neq 0$, integrating both sides give

$$x^c y = \frac{C_1 x^{-b+1}}{-b+1} + C_2$$

for some (real) constant $C_2$.
Hence,

$$y = x^{-c}\left(\frac{C_1 x^{-b+1}}{-b+1} + C_2\right)$$

for some (real) constant $C_1, C_2$.
Hence,

$$\begin{aligned} y &= x^{-(-p\pm q)}\left(\frac{C_1 x^{-(\mp 2q+1)+1}}{-(\mp 2q+1)+1} + C_2\right) \\ &= x^{p\mp q}\left(\frac{C_1 x^{\pm 2q}}{\pm 2q} + C_2\right). \\ &= \frac{C_1}{\pm 2q} x^{p\pm q} + C_2 x^{p\mp q} \\ &= C_3 x^{p\pm q} + C_2 x^{p\mp q}, \end{aligned}$$

for some (real) constant $C_2, C_3$.

2. This is when $q = 0$ and $f(x) = x^n$. We have $a = p+1, b = 1$ and $c = -p$, and the original differential equation reduces to

$$x^{p+1}\left(x\left(x^{-p}y\right)'\right)' = x^n,$$

and hence

$$\left(x\left(x^{-p}y\right)'\right)' = x^{n-p-1}.$$

There are two cases here:

(a) If $n - p - 1 = -1$, i.e. $n = p$, we have, by integration,

$$x\left(x^{-p}y\right)' = \ln x + C_1.$$

This gives

$$\left(x^{-p}y\right)' = \frac{\ln x}{x} + \frac{C_1}{x},$$

and hence by integration

$$x^{-p}y = \frac{(\ln x)^2}{2} + C_1 \ln x + C_2.$$

This solves to

$$y = \frac{x^p(\ln x)^2}{2} + C_1 x^p \ln x + C_2 x^p.$$

(b) If $n - p - 1 \neq -1$, i.e. $n \neq p$, we have

$$x\left(x^{-p}y\right)' = \frac{x^{n-p}}{n-p} + C_1.$$

This gives

$$\left(x^{-p}y\right)' = \frac{x^{n-p-1}}{n-p} + \frac{C_1}{x}.$$

Since $n - p - 1 \neq -1$, by integration we have

$$x^{-p}y = \frac{x^{n-p}}{(n-p)^2} + C_1 \ln x + C_2,$$

and hence

$$y = \frac{x^n}{(n-p)^2} + C_1 x^p \ln x + C_2 x^p.$$