2018.3 Question 2

1. Notice that

$$\begin{aligned}\frac{\mathrm{d}y_n}{\mathrm{d}x} &= \frac{\mathrm{d}(-1)^n \frac{1}{z}\frac{\mathrm{d}^n z}{\mathrm{d}x^n}}{\mathrm{d}x} \\ &= (-1)^n \left[\frac{\mathrm{d}\frac{1}{z}}{\mathrm{d}x}\cdot\frac{\mathrm{d}^n z}{\mathrm{d}x^n} + \frac{1}{z}\cdot\frac{\mathrm{d}\frac{\mathrm{d}^n z}{\mathrm{d}x^n}}{\mathrm{d}x}\right] \\ &= (-1)^n \left[\frac{2x}{z}\cdot\frac{\mathrm{d}^n z}{\mathrm{d}x^n} + \frac{1}{z}\cdot\frac{\mathrm{d}^{n+1} z}{\mathrm{d}x^{n+1}}\right] \\ &= 2x\cdot(-1)^n\frac{1}{z}\frac{\mathrm{d}^n z}{\mathrm{d}x^n} - (-1)^{n+1}\frac{1}{z}\frac{\mathrm{d}^{n+1} z}{\mathrm{d}x^{n+1}} \\ &= 2xy_n - y_{n+1},\end{aligned}$$

as desired.

2. We first look at the base case where $n = 1$. What is desired is

$$y_2 = 2xy_1 - 2y_0.$$

We have $y_0 = 1$,

$$y_1 = (-1)^1 \frac{1}{e^{-x^2}} \frac{\mathrm{d}e^{-x^2}}{\mathrm{d}x} = -e^{x^2}(-2x)e^{-x^2} = 2x,$$

and

$$y_2 = 2xy_1 - \frac{\mathrm{d}y_1}{\mathrm{d}x} = 2x\cdot 2x - 2 = 4x^2 - 2.$$

Hence,

$$2xy_1 - 2y_0 = 2x\cdot 2x - 2\cdot 1 - 4x^2 - 2 = y_2,$$

so the base case is satisfied.

Now assume this is true for some $n = k \geq 1$, i.e.

$$y_{k+1} = 2xy_k - 2ky_{k-1}.$$

We have

$$\begin{aligned}y_{k+2} &= 2xy_{k+1} - \frac{\mathrm{d}y_{k+1}}{\mathrm{d}x} \\ &= 2xy_{k+1} - \frac{\mathrm{d}\left(2xy_k - 2ky_{k-1}\right)}{\mathrm{d}x} \\ &= 2xy_{k+1} - 2y_k - 2x\frac{\mathrm{d}y_k}{\mathrm{d}x} + 2k\frac{\mathrm{d}y_{k-1}}{\mathrm{d}x} \\ &= 2xy_{k+1} - 2y_k - 2x(2xy_k - y_{k+1}) + 2k(2xy_{k-1} - y_k) \\ &= 2xy_{k+1} - 2y_k - 4x^2y_k + 2xy_{k+1} + 4kxy_{k-1} - 2ky_k \\ &= 4xy_{k+1} - 2(2x^2 + k + 1)y_k + 4kxy_{k-1} \\ &= 4xy_{k+1} - 2(2x^2 + k + 1)y_k + 4kx\cdot\frac{2xy_k - y_{k+1}}{2k} \\ &= 4xy_{k+1} - 2(2x^2 + k + 1)y_k + 2x(2xy_k - y_{k+1}) \\ &= 2xy_{k+1} - 2(k+1)y_k,\end{aligned}$$

which is exactly the statement for $n = k + 1$.

Hence, by the principle of mathematical induction, we have $y_{n+1} = 2xy_n - 2ny_{n-1}$ for all $n \geq 1$.

We have

$$\begin{aligned}\text{LHS} &= y_{n+1}^2 - y_n y_{n+2} \\ &= y_{n+1}^2 - y_n(2xy_{n+1} - 2(n+1)y_n) \\ &= y_{n+1}^2 - 2xy_n y_{n+1} + 2(n+1)y_n^2\end{aligned}$$

and

$$\begin{aligned}
\text{RHS} &= 2n(y_n^2 - y_{n-1}y_{n+1}) + 2y_n^2 \\
&= 2n\left(y_n^2 - \frac{2xy_n - y_{n+1}}{2n}y_{n+1}\right) + 2y_n^2 \\
&= 2ny_n^2 - (2xy_n - y_{n+1})\,y_{n+1} + 2y_n^2 \\
&= 2ny_n^2 - 2xy_ny_{n+1} + y_{n+1}^2 + 2y_n^2 \\
&= y_{n+1}^2 - 2xy_ny_{n+1} + 2(n+1)y_n^2.
\end{aligned}$$

3. This can be shown by induction on n. The base case for $n = 1$ is

$$y_1^2 - y_0y_2 = (2x)^2 - 1 \cdot (4x^2 - 2) = 2 > 0$$

is true.

Now assume the statement is true for $n = k \geq 1$, i.e.

$$y_k^2 - y_{k-1}y_{k+1} > 0.$$

We have

$$\begin{aligned}
y_{k+1}^2 - y_ky_{k+2} &= 2n(y_k^2 - y_{k-1}y_k + 1) + 2y_n^2 \\
&> 2n \cdot 0 + y_n^2 \\
&= 0 + y_n^2 \\
&\geq 0,
\end{aligned}$$

which is the statement for $n = k + 1$.

Hence, by the principle of mathematical induction, we have $y_n^2 - y_{n-1}y_{n+1} > 0$ for all $n \geq 1$.