2018.3 Question 13

By the definition of a probability generating function, we have

$$G(1) = \sum_{n=0}^{\infty} \mathrm{P}(X = n), \text{ and } G(-1) = \sum_{n=0}^{\infty} (-1)^n \mathrm{P}(X = n).$$

Hence,

$$G(1) + G(-1) = \sum_{n=0}^{\infty} [1 + (-1)^n] \mathrm{P}(X = n).$$

When n is odd, $1 + (-1)^n = 0$. When n is even, $1 + (-1)^n = 2$.
This means

$$G(1) + G(-1) = 2 \sum_{n=0}^{\infty} \mathrm{P}(X = 2n),$$

which gives

$$\frac{1}{2}(G(1) + G(-1)) = \sum_{n=0}^{\infty} \mathrm{P}(X = 2n) = \mathrm{P}(X = 0 \text{ or } X = 2 \text{ or } X = 4 \ldots).$$

Since $X \sim \mathrm{Po}(\lambda)$, we have

$$\mathrm{P}(X = x) = e^{-\lambda} \frac{\lambda^x}{x!},$$

and hence the probability generating function for X, $G(t)$, must satisfy

$$\begin{aligned} G(t) &= \sum_{n=0}^{\infty} \mathrm{P}(X = n) \cdot t^n \\ &= \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \cdot t^n \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \\ &= e^{-\lambda} \cdot e^{\lambda t} \\ &= e^{-\lambda(1-t)}. \end{aligned}$$

1. Consider $G(t) + G(-t)$. By definition, we have

$$G(t) = \sum_{n=0}^{\infty} \mathrm{P}(X = n) t^n, G(-t) = \sum_{n=0}^{\infty} (-1)^n \mathrm{P}(X = n) t^n,$$

and hence

$$G(t) + G(-t) = \sum_{n=0}^{\infty} (1 + (-1)^n) \mathrm{P}(X = n) t^n = 2 \sum_{n=0}^{\infty} \mathrm{P}(X = 2n) t^{2n}.$$

Let $H(t)$ be the probability generating function of Y, we have

$$\begin{aligned} H(t) &= \sum_{n=0}^{\infty} \mathrm{P}(Y = n) \cdot t^n \\ &= \sum_{n=0}^{\infty} \mathrm{P}(Y = 2n) \cdot t^{2n} \\ &= \sum_{n=0}^{\infty} k \, \mathrm{P}(X = 2n) \cdot t^{2n} \\ &= \frac{k}{2} \left(G(t) + G(-t) \right). \end{aligned}$$

To find k, we must have $H(1) = 1$. Hence,

$$1 = \frac{k}{2}\left(G(1) + G(-1)\right) = \frac{k}{2}\left(e^{-\lambda(1-1)} + e^{-\lambda(1+1)}\right) = \frac{k}{2}\left(1 + e^{-2\lambda}\right),$$

which gives

$$k = \frac{2}{1 + e^{-2\lambda}} = \frac{2e^{\lambda}}{e^{\lambda} + e^{-\lambda}} = \frac{e^{\lambda}}{\cosh\lambda}.$$

Hence,

$$\begin{aligned} H(t) &= \frac{k}{2}\left(G(t) + G(-t)\right) \\ &= \frac{e^{\lambda}}{2\cosh\lambda}\left(e^{-\lambda(1-t)} + e^{-\lambda(1+t)}\right) \\ &= \frac{1}{\cosh\lambda}\frac{e^{\lambda t} + e^{-\lambda t}}{2} \\ &= \frac{\cosh\lambda t}{\cosh\lambda}. \end{aligned}$$

Differentiating this with respect to t, we have

$$H'(t) = \frac{\lambda\sinh\lambda t}{\cosh\lambda},$$

and hence

$$\mathrm{E}(Y) = H'(1) = \frac{\lambda\sinh\lambda\cdot 1}{\cosh\lambda} = \lambda\tanh\lambda.$$

Since $-1 < \tanh\lambda < 1$, we have $\lambda\tanh\lambda < \lambda$, and so $\mathrm{E}(Y) < \lambda$ for $\lambda > 0$.

2. Consider $G(t) + G(-t) + G(it) + G(-it)$. By definition, we have

$$G(t) + G(-t) + G(it) + G(-it) = \sum_{n=0}^{\infty}\left(1 + (-1)^n + i^n + (-i)^n\right)\mathrm{P}(X = n)\cdot t^n.$$

Let m be an integer. Consider the following four cases:

- $n = 4m$, $1 + (-1)^n + i^n + (-i)^n = 1 + 1 + 1 + 1 = 4$.
- $n = 4m + 1$, $1 + (-1)^n + i^n + (-i)^n = 1 + (-1) + i + (-i) = 0$.
- $n = 4m + 2$, $1 + (-1)^n + i^n + (-i)^n = 1 + 1 + (-1) + (-1) = 0$.
- $n = 4m + 3$, $1 + (-1)^n + i^n + (-i)^n + 1 + (-1) + (-i) + i = 0$.

Hence,

$$G(t) + G(-t) + G(it) + G(-it) = 4\sum_{n=0}^{\infty}\mathrm{P}(X = 4n)\cdot t^{4n}.$$

Let $P(t)$ be the probability generating function of Z, we have

$$\begin{aligned} P(t) &= \sum_{n=0}^{\infty}\mathrm{P}(Z = n)\cdot t^n \\ &= \sum_{n=0}^{\infty}\mathrm{P}(Z = 4n)\cdot t^{4n} \\ &= c\sum_{n=0}^{\infty}\mathrm{P}(X = 4n)\cdot t^{4n} \\ &= \frac{c}{4}\left(G(t) + G(-t) + G(it) + G(-it)\right). \end{aligned}$$

Since $P(1) = 0$, we must have

$$\begin{aligned}1 &= \frac{c}{4}\left(G(1) + G(-1) + G(i) + G(-i)\right) \\ &= \frac{c}{4}\left(e^{-\lambda(1-1)} + e^{-\lambda(1+1)} + e^{-\lambda(1-i)} + e^{-\lambda(1+i)}\right) \\ &= \frac{ce^{-\lambda}}{4}\left(e^{\lambda} + e^{-\lambda} + e^{i\lambda} + e^{-i\lambda}\right) \\ &= \frac{ce^{-\lambda}}{2}\left(\cos\lambda + \cosh\lambda\right).\end{aligned}$$

Hence,

$$c = \frac{2e^{\lambda}}{\cos\lambda + \cosh\lambda}.$$

Therefore,

$$\begin{aligned}P(t) &= \frac{c}{4}\left(G(t) + G(-t) + G(it) + G(-it)\right) \\ &= \frac{e^{\lambda}}{2(\cos\lambda + \cosh\lambda)}\left[e^{-\lambda(1-t)} + e^{-\lambda(1+t)} + e^{-\lambda(1-it)} + e^{-\lambda(1+it)}\right] \\ &= \frac{e^{\lambda t} + e^{-\lambda t} + e^{\lambda it} + e^{-\lambda it}}{2(\cos\lambda + \cosh\lambda)} \\ &= \frac{\cos\lambda t + \cosh\lambda t}{\cos\lambda + \cosh\lambda}.\end{aligned}$$

Differentiating this with respect to t gives us

$$P'(t) = \frac{\lambda(-\sin\lambda t + \sinh\lambda t)}{\cos\lambda + \cosh\lambda},$$

and hence

$$\mathrm{E}(Z) = P'(1) = \frac{\lambda(-\sin\lambda + \sinh\lambda)}{\cos\lambda + \cosh\lambda}.$$

$\mathrm{E}(Z) < \lambda$ is equivalent to

$$\frac{\sinh\lambda - \sin\lambda}{\cosh\lambda + \cos\lambda} < 1,$$

which is then equivalent to

$$\sinh\lambda - \cosh\lambda < \sin\lambda + \cos\lambda,$$

which is

$$-e^{-\lambda} < \sin\lambda + \cos\lambda.$$

However, this is not necessarily true. Let $\lambda = \pi$. We have

$$\text{LHS} = -e^{-\pi} > -e^{0} = -1,$$

and

$$\text{RHS} = \sin\pi + \cos\pi = -1,$$

which means LHS > RHS for $\lambda = \pi$, which means $\mathrm{E}(Z) > \lambda$. Therefore, the statement is not true.