2018.3 Question 12

1. $\mathrm{P}(Y_k) \leq y$ is the probability that there is at least k numbers that are less than equal to y.

   If there are $k \leq m \leq n$ numbers less than or equal to y, then there must be $n-m$ numbers greater than or equal to y. The probability of the first thing happening for each number is y, and for the second thing happening for each number is $1-y$. We also have to choose m numbers from the n to make them less than or equal to y. Therefore,

   $$\mathrm{P}(Y_k \leq y) = \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}.$$

2. We have

   $$m\binom{n}{m} = m \cdot \frac{n!}{m!(n-m)!} = \frac{n!}{(m-1)!(n-m)!} = n \cdot \frac{(n-1)!}{(m-1)!(n-m)!} = n\binom{n-1}{m-1}.$$

   We have

   $$(n-m)\binom{n}{m} = (n-m) \cdot \frac{n!}{m!(n-m)!} = \frac{n!}{m!(n-m-1)!} = n \cdot \frac{(n-1)!}{m!(n-m-1)!} = n\binom{n-1}{m}.$$

   The cumulative distribution function F_{Y_k} is

   $$F_{Y_k}(y) = \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}.$$

   Therefore, the probability density function f_{Y_k} is

   $$\begin{aligned}
   f_{Y_k}(y) &= F'_{Y_k}(y) \\
   &= \sum_{m=k}^{n} \binom{n}{m} \left[m y^{m-1}(1-y)^{n-m} - (n-m) y^m (1-y)^{n-m-1} \right] \\
   &= \sum_{m=k}^{n} y^{m-1}(1-y)^{n-m-1} \left[m\binom{n}{m}(1-y) - (n-m)\binom{n}{m} y \right] \\
   &= n \left[\sum_{m=k}^{n} \binom{n-1}{m-1} y^{m-1}(1-y)^{n-m} - \sum_{m=k}^{n-1} \binom{n-1}{m} y^m (1-y)^{n-m-1} \right] \\
   &= n \left[\sum_{m=k}^{n} \binom{n-1}{m-1} y^{m-1}(1-y)^{n-m} - \sum_{m=k+1}^{n} \binom{n-1}{m-1} y^{m-1} (1-y)^{n-m} \right] \\
   &= n\binom{n-1}{k-1} y^{k-1}(1-y)^{n-k}.
   \end{aligned}$$

   Since $Y_k \in [0,1]$, we must have

   $$\int_0^1 f_{Y_k}(y)\,\mathrm{d}y = 1,$$

   and hence

   $$n\binom{n-1}{k-1} \int_0^1 y^{k-1}(1-y)^{n-k}\,\mathrm{d}y = 1,$$

   and therefore we have

   $$\int_0^1 y^{k-1}(1-y)^{n-k}\,\mathrm{d}y = \frac{1}{n\binom{n-1}{k-1}}.$$

3. By the definition of the expectation,

$$\begin{aligned}
\mathrm{E}(Y_k) &= \int_0^1 y f_{Y_k}(y) \, \mathrm{d}y \\
&= n \binom{n-1}{k-1} \int_0^1 y^k (1-y)^{n-k} \, \mathrm{d}y \\
&= n \binom{n-1}{k-1} \cdot \frac{1}{(n+1)\binom{n}{k}} \\
&= \frac{n \cdot \frac{(n-1)!}{(k-1)!(n-k)!}}{(n+1) \cdot \frac{n!}{k!(n-k)!}} \\
&= \frac{\frac{n!}{(k-1)!(n-k)!}}{\frac{(n+1)n!}{k(k-1)!(n-k)!}} \\
&= \frac{k}{n+1}.
\end{aligned}$$