2018.3 Question 1

1. By differentiation with respect to β, we have

$$f'(\beta) = 1 + \frac{1}{\beta^2} + \frac{2}{\beta^3}.$$

If $f'(t) = 0$, we must have

$$t^3 + t + 2 = 0.$$

Therefore,

$$(t + 1)(t^2 - t + 2) = 0,$$

and hence the only real root to this is $t = -1$, since $(-1)^2 - 2 \cdot 4 < 0$.

This means the only stationary point of $y = f(\beta)$ is $(-1, f(-1) = -1)$.

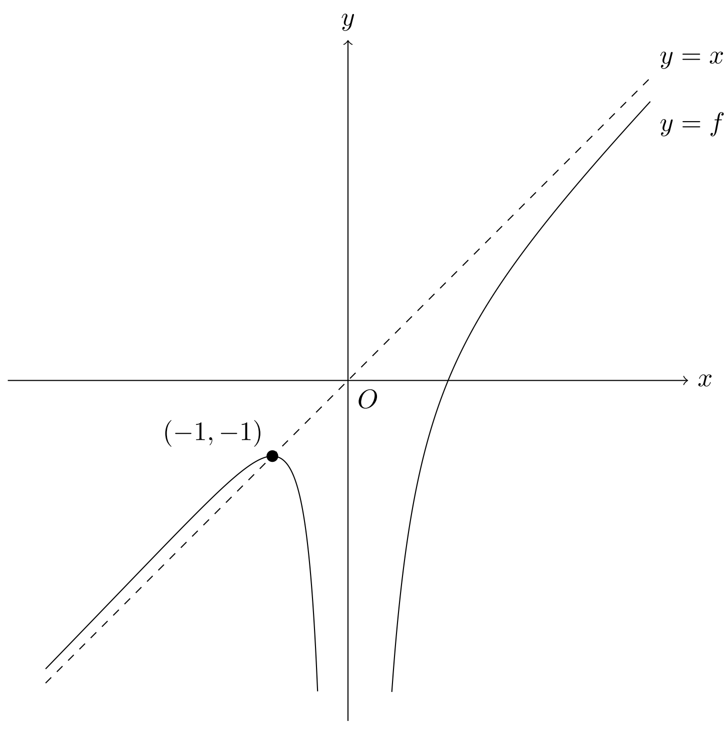
For the limiting behaviour of the function, we first look at the case where $\beta > 0$. As $\beta \to \infty$, we have $f(\beta) \to \beta$ from below. As $\beta \to 0^+$, we have $f(\beta) \to -\frac{1}{\beta} - \frac{1}{\beta^2} \to -\infty$.

When $\beta < 0$, we use the substitution $t = -\frac{1}{\beta}$ to make the behaviours more convincing, and hence

$$f(\beta) = \beta + t - t^2.$$

As $\beta \to 0^-$, we have $t \to \infty$, and $f(\beta) \to t - t^2 \to -\infty$. As $\beta \to -\infty$, we have $t \to 0^+$, and $f(\beta) \to \beta$ from above, since $t - t^2 = t(1 - t) > 0$ when $0 < t < 1$.

This means the curve $y = f(\beta)$ is as below.



Similarly, by differentiation with respect to β, we have

$$g'(\beta) = 1 - \frac{3}{\beta^2} + \frac{2}{\beta^3}.$$

If $g'(t) = 0$, we must have

$$t^3 - 3t + 2 = 0.$$

Therefore,

$$(t - 1)^2(t + 2) = 0,$$

and hence the real roots to this is $t = 1$ and $t = -2$.

This means the stationary points of $y = g(\beta)$ is $(1, g(1) = 3)$ and $(-2, g(-2) = -\frac{15}{4})$.

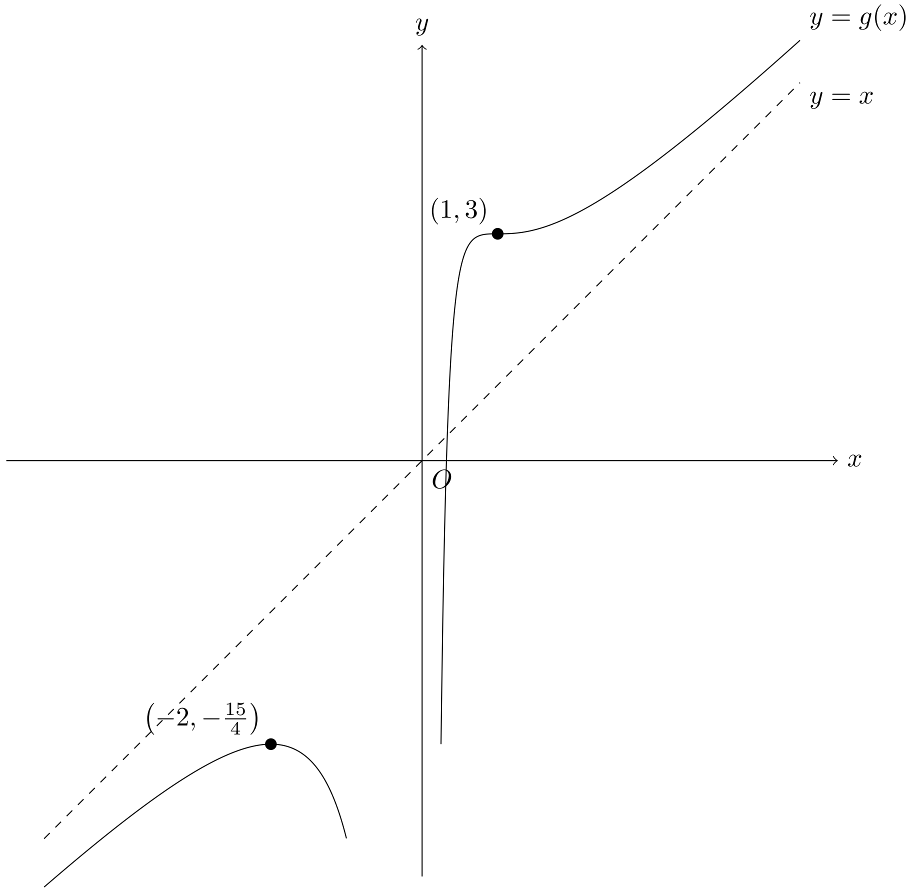
For the limiting behaviour of the function, we first look at the case where $\beta > 0$. We consider the substitution $t = -\frac{1}{\beta}$ to make the behaviours more convincing, and hence

$$g(\beta) = \beta - 3t - t^2.$$

As $\beta \to \infty$, $t \to 0^-$, and hence $f(\beta) \to \beta$ from below, since $-3t - t^2 = -t(t+3) > 0$ for $-3 < t < 0$. As $\beta \to 0^+$, $t \to -\infty$, and hence $f(\beta) \to -3t - t^2 \to -\infty$.

When $\beta < 0$, we have as $\beta \to 0^-$, $f(\beta) \to -\infty$. As $\beta \to -\infty$, $f(\beta) \to \beta$ from below.

This means the curve $y = g(\beta)$ is as below.



2. By Vieta's Theorem, we have $u + v = -\alpha$, and $uv = \beta$. Hence,

$$u + v + \frac{1}{uv} = -\alpha + \frac{1}{\beta},$$

and

$$\frac{1}{u} + \frac{1}{v} + uv = \frac{u+v}{uv} + uv = -\frac{\alpha}{\beta} + \beta.$$

3. By the given condition, we have

$$-\alpha + \frac{1}{\beta} = -1 \iff \alpha = 1 + \frac{1}{\beta}.$$

Hence,

$$\begin{aligned}\frac{1}{u}+\frac{1}{v}+uv &= -\frac{\alpha}{\beta}+\beta\\ &= -\frac{1+\frac{1}{\beta}}{\beta}+\beta\\ &= \frac{\beta^2-1-\frac{1}{\beta}}{\beta}\\ &= \beta-\frac{1}{\beta}-\frac{1}{\beta^2}\\ &= f(\beta).\end{aligned}$$

Also, since u, v are both real, we have

$$\begin{aligned}\alpha^2-4\beta &= \left(1+\frac{1}{\beta}\right)^2-4\beta\\ &= 1+\frac{2}{\beta}+\frac{1}{\beta^2}-4\beta\\ &= \frac{-4\beta^3+\beta^2+2\beta+1}{\beta^2}\\ &\geq 0.\end{aligned}$$

Multiplying both sides by $-\beta^2$ (which flips the sign) gives

$$\begin{aligned}4\beta^3-\beta^2-2\beta-1 &\leq 0\\ (\beta-1)(4\beta^2+3\beta+1) &\leq 0.\end{aligned}$$

This cubic has exactly one real root $\beta = 1$, so the solution to this inequality is $\beta \leq 1$ and $\beta \neq 0$. Notice that f is increasing on $(0, 1] \subset (0, \infty)$. Therefore, for $\beta > 0$,

$$f(\beta) \leq f(1) = 1-1-1 = -1.$$

When $\beta < 0$, we have

$$f(\beta) \leq f(-1) = -1.$$

So for the range of β in this question, we always have $f(\beta) \leq -1$. But we also have $\frac{1}{u}+\frac{1}{v}+uv \leq -1$ as shown before. These gives us exactly our desired statement.

4. By the given condition, we have

$$-\alpha+\frac{1}{\beta} = 3 \iff \alpha = -3+\frac{1}{\beta}.$$

Hence,

$$\begin{aligned}\frac{1}{u}+\frac{1}{v}+uv &= -\frac{\alpha}{\beta}+\beta\\ &= -\frac{-3+\frac{1}{\beta}}{\beta}+\beta\\ &= \beta+\frac{3}{\beta}-\frac{1}{\beta^2}\\ &= g(\beta).\end{aligned}$$

Also, since u, v are both real, we have $\beta \leq 1$ and $\beta \neq 0$ as well.

g must be increasing on $(0, 1]$. Hence, for $\beta > 0$, we have

$$g(\beta) \leq g(1) = 3.$$

When $\beta < 0$, we have

$$g(\beta) \leq g(-2) = -\frac{15}{4}.$$

Since $3 > -\frac{15}{4}$, we can conclude that the maximum value of $\frac{1}{u} + \frac{1}{v} + uv$ is 3, and it is taken when $\beta = 1$, which corresponds to $\alpha = -2$.