2018.2 Question 8

1. Since $v = \sqrt{y}$, we have $y = v^2$, and hence
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2v\frac{\mathrm{d}v}{\mathrm{d}t},$$
and hence the original equation reduces to
$$2v\frac{\mathrm{d}v}{\mathrm{d}t} = \alpha v - \beta v^2,$$
which gives
$$2\frac{\mathrm{d}v}{\mathrm{d}t} = \alpha - \beta v.$$
Rearranging gives us
$$\frac{\mathrm{d}v}{\alpha - \beta v} = \frac{\mathrm{d}t}{2},$$
and hence integrating both sides gives
$$-\frac{1}{\beta}\ln|\alpha - \beta v| = \frac{1}{2}t + C.$$
Hence,
$$\ln|\alpha - \beta v| = -\frac{\beta t}{2} + C',$$
and
$$\alpha - \beta v = A\exp\left(-\frac{\beta t}{2}\right),$$
and hence
$$v = \frac{1}{\beta}\left[\alpha + A\exp\left(-\frac{\beta}{2}\right)\right],$$
which means
$$y = v^2 = \frac{1}{\beta^2}\left[\alpha + A\exp\left(-\frac{\beta t}{2}\right)\right]^2.$$
Since $y = 0$ when $t = 0$, we have $A = -\alpha$, and hence
$$y_1(t) = \frac{\alpha^2}{\beta^2}\left[1 - \exp\left(-\frac{\beta t}{2}\right)\right]^2.$$

2. Let $v = \sqrt[3]{y}$ in this case, and hence $y = v^3$, we have
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3v^2\frac{\mathrm{d}v}{\mathrm{d}t},$$
and hence the original equation reduces to
$$3v^2\frac{\mathrm{d}v}{\mathrm{d}t} = \alpha v^2 - \beta v^3,$$
and hence
$$3\frac{\mathrm{d}v}{\mathrm{d}t} = \alpha - \beta v.$$
Similar to before, this solves to
$$v = \frac{1}{\beta}\left[\alpha + B\exp\left(-\frac{\beta}{3}\right)\right],$$
and hence
$$y = v^3 = \frac{1}{\beta^3}\left[\alpha + B\exp\left(-\frac{\beta}{3}\right)\right]^3.$$
Since $y = 0$ when $t = 0$, we have $B = -\alpha$, and hence
$$y_2(t) = \frac{\alpha^3}{\beta^3}\left[1 - \exp\left(-\frac{\beta t}{3}\right)\right]^3.$$

3. Let $\alpha = \beta = \gamma$. We have

$$y_1(t) = \left[1 - \exp\left(-\frac{\gamma t}{2}\right)\right]^2, y_2(t) = \left[1 - \exp\left(-\frac{\gamma t}{3}\right)\right]^3.$$

For $t > 0$, we have

$$0 > -\frac{\gamma t}{3} > -\frac{\gamma t}{2} > -\infty,$$

and since the exponential function is strictly increasing, we have

$$1 > \exp\left(-\frac{\gamma t}{3}\right) > \exp\left(-\frac{\gamma t}{2}\right) > 0,$$

and hence

$$1 > 1 - \exp\left(-\frac{\gamma t}{2}\right) > 1 - \exp\left(-\frac{\gamma t}{3}\right) > 0.$$

Hence,

$$y_1(t) = \left[1 - \exp\left(-\frac{\gamma t}{2}\right)\right]^2 > \left[1 - \exp\left(-\frac{\gamma t}{3}\right)\right]^2 > \left[1 - \exp\left(-\frac{\gamma t}{3}\right)\right]^3 = y_2(t)$$

which tells us that the graph of y_2 should lie below the graph of y_1.
As $t \to \infty$,

$$\exp\left(-\frac{\gamma t}{2}\right), \exp\left(-\frac{\gamma t}{2}\right) \to 0^+,$$

and hence

$$y_1(t), y_2(t) \to 1^-.$$

At $t = 0$, $y_1(t) = y_2(t) = 0$, and hence by the original differential equation $y_1'(t) = y_2'(t) = 0$.
Hence, the graph looks as follows.