2018.2 Question 7

Since $|MQ| = \mu|QB|$, we must have $|MQ| = \frac{\mu}{1+\mu}|MB|$, and hence

$$\overrightarrow{MQ} = \frac{\mu}{1+\mu}\overrightarrow{MB},$$

and hence

$$\mathbf{q} - \mathbf{m} = \frac{\mu}{1+\mu}\left(\mathbf{b} - \mathbf{m}\right).$$

Similarly,

$$\mathbf{q} - \mathbf{n} = \frac{\nu}{1+\nu}\left(\mathbf{a} - \mathbf{n}\right).$$

Since $\mathbf{q} = \mathbf{q}$, we have

$$\frac{\mu}{1+\mu}\left(\mathbf{b} - \mathbf{m}\right) + \mathbf{m} = \frac{\nu}{1+\nu}\left(\mathbf{a} - \mathbf{n}\right) + \mathbf{n},$$

which rearranges to give

$$\frac{1}{1+\mu}\mathbf{m} - \frac{1}{1+\nu}\mathbf{n} = \frac{\nu}{1+\nu}\mathbf{a} - \frac{\mu}{1+\mu}\mathbf{b}.$$

Since $\mathbf{m}$ is a scalar multiple of $\mathbf{a}$ as M is on the side OA, and $\mathbf{n}$ is a scalar multiple of $\mathbf{b}$ similarly, and $\mathbf{a}$ and $\mathbf{b}$ are linearly independent since OAB forms a triangle, we can conclude that

$$\mathbf{m} = \frac{1+\mu}{1} \cdot \frac{\nu}{1+\nu}\mathbf{a},$$

and hence

$$\mathbf{m} = \frac{(1+\mu)\nu}{1+\nu}\mathbf{a}.$$

Similarly,

$$\mathbf{n} = \frac{(1+\nu)\mu}{1+\mu}\mathbf{b}.$$

Since L lies on OB with $|OL| = \lambda|OB|$, then we have

$$\mathbf{l} = \lambda\mathbf{b},$$

and hence

$$\overrightarrow{ML} = \mathbf{l} - \mathbf{m} = \lambda\mathbf{b} - \frac{(1+\mu)\nu}{1+\nu}\mathbf{a}.$$

Since

$$\overrightarrow{AN} = \mathbf{n} - \mathbf{a} = \frac{(1+\nu)\mu}{1+\mu}\mathbf{b} - \mathbf{a}.$$

$\overrightarrow{ML}$ is parallel to $\overrightarrow{AN}$ means that the corresponding scalar vectors for $\mathbf{a}$ and $\mathbf{b}$ are in ratio (since they are linearly independent), and hence

$$\lambda : \frac{(1+\nu)\mu}{1+\mu} = \left(-\frac{(1+\mu)\nu}{1+\nu}\right) : (-1),$$

and hence

$$\lambda = \frac{(1+\mu)\nu}{1+\nu} \cdot \frac{(1+\nu)\mu}{1+\mu} = \mu\nu.$$

The condition $\mu\nu < 1$ ensured that L lies on OB between O and B (i.e. on the side OB).