2018.2 Question 6

1. Notice that for $n \geq 5$, $n! = 5 \cdot 4! \cdot 6 \cdot 7 \cdots n$, and $n! = 5k$ for $k = 4! \cdot 6 \cdot 7 \cdots n > 1$ is an integer.

   Therefore,
   $$n! + 5 = 5k + 5 = 5(k+1)$$
   is a multiple of two integers greater than 1, and hence p cannot be prime.

   Hence, $n < 5$.

   If $n = 1$, $n! + 5 = 6$ is not prime.

   If $n = 2$, $n! + 5 = 7$ is prime. $(n, p) = (2, 7)$ is a solution.

   If $n = 3$, $n! + 5 = 11$ is prime. $(n, p) = (3, 11)$ is a solution.

   If $n = 4$, $n! + 5 = 29$ is prime. $(n, p) = (4, 29)$ is a solution.

   Therefore, all solutions are $(n, p) = (2, 7), (3, 11)$ and $(4, 29)$.

2. If $n \geq 7$, then we have
   $$m! = 1! \times 3! \times \cdots \times (2n-1)! > (4n)!$$
   and hence $m > 4n$.

   Let p be some prime number between $2n$ and $4n$. Therefore, $m!$ must include p as one of its terms, and $p \mid m! = \text{RHS}$.

   However, on the left-hand side, all the terms are less than p, and since p is a prime, it must not divide any term in the left-hand side factorial expansion (since every term in the expansion is less than p), and hence $p \nmid \text{LHS}$.

   But since LHS = RHS this is impossible, and we can deduce that $n < 7$.

   - $n = 1$, $\text{LHS} = 1! = 1$ and $(n, m) = (1, 1)$ is a solution.
   - $n = 2$, $\text{LHS} = 1! \cdot 3! = 3!$ and $(n, m) = (2, 3)$ is a solution.
   - $n = 3$, $\text{LHS} = 1! \cdot 3! \cdot 5! = 6 \cdot 5! = 6!$ and $(n, m) = (3, 6)$ is a solution.
   - $n = 4$, $\text{LHS} = 1! \cdot 3! \cdot 5! \cdot 7! = 6! \cdot 7! = 7! \cdot 6! = 7! \cdot (3 \cdot 2 \cdot 5 \cdot 4 \cdot 3 \cdot 2) = 7! \cdot (2 \cdot 4) \cdot (3 \cdot 3) \cdot (2 \cdot 5) = 10!$ and $(n, m) = (4, 10)$ is a solution.
   - $n = 5$, $\text{LHS} = 1! \cdot 3! \cdot 5! \cdot 7! \cdot 9! = 10! \cdot 9! > 10!$, so if m exists, $m > 10$ and $m \geq 11$. Then $11 \mid \text{RHS} = \text{LHS}$, but this is impossible since $11 > 9$, so such m does not exist.
   - $n = 6$, $\text{LHS} = 1! \cdot 3! \cdot 5! \cdot 7! \cdot 9! \cdot 11! = 10! \cdot 9! \cdot 11! = 11! \cdot 9! \cdot 10! = 12! \cdot 10! \cdot (9 \cdot 8 \cdot 7 \cdot 5 \cdot 4 \cdot 3) > 12!$, so if m exists, $m > 12$ and $m \geq 13$. Then $13 \mid \text{RHS} = \text{LHS}$, but this is impossible since $13 > 11$, and so such m does not exist.

   Hence, the only possible solutions are
   $$(n, m) \in \{(1, 1), (2, 3), (3, 6), (4, 10)\}.$$