2018.2 Question 5

1. For $|x| < 1$, we have
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{t=0}^{\infty} (-x)^t.$$
Since $\ln(1+x)$ differentiates to $\frac{1}{1+x}$, by integration, we have
$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x} \, \mathrm{d}x \\ &= \int \sum_{t=0}^{\infty} (-x)^t \, \mathrm{d}x \\ &= \sum_{t=0}^{\infty} (-1)^t \int x^t \, \mathrm{d}x \\ &= C + \sum_{t=0}^{\infty} (-1)^t \frac{x^{t+1}}{t+1} \\ &= C - \sum_{t=1}^{\infty} \frac{(-x)^t}{t}.\end{aligned}$$
Let $x = 0$, and we see $\ln(1+x) = \ln 1 = 0$, and the sum on the right-hand side evaluates to 0, and hence $C = 0$. This gives the Maclaurin expansion for $\ln(1+x)$
$$\ln(1+x) = -\sum_{t=1}^{\infty} \frac{(-x)^t}{t}.$$

2. We have
$$e^{-ax} = \sum_{t=0}^{\infty} \frac{(-ax)^t}{t!},$$
and hence
$$\begin{aligned}&\int_0^{\infty} \frac{(1 - e^{-ax})e^{-x}}{x} \, \mathrm{d}x \\ = &\int_0^{\infty} \frac{-\sum_{t=1}^{\infty} \frac{(-ax)^t}{t!} \cdot e^{-x}}{x} \, \mathrm{d}x \\ = &\sum_{t=1}^{\infty} \int_0^{\infty} \frac{-(-ax)^t e^{-x}}{t! x} \, \mathrm{d}x \\ = &\sum_{t=1}^{\infty} \int_0^{\infty} \frac{(-x)^{t-1} a^t e^{-x}}{t!} \, \mathrm{d}x \\ = &\sum_{t=1}^{\infty} \frac{(-1)^{t-1} a^t}{t!} \int_0^{\infty} x^{t-1} e^{-x} \, \mathrm{d}x.\end{aligned}$$
We aim to find an expression for
$$I_t = \int_0^{\infty} x^t e^{-x} \, \mathrm{d}x.$$

Using integration by parts, we have

$$\begin{aligned} I_t &= \int_0^\infty x^t e^{-x} \, \mathrm{d}x \\ &= -\int_0^\infty x^t \, \mathrm{d}e^{-x} \\ &= -\left[(x^t e^{-x})_0^\infty - \int_0^\infty e^{-x} \, \mathrm{d}x^t \right] \\ &= t \int_0^\infty e^{-x} x^{t-1} \, \mathrm{d}x \\ &= t I_{t-1}, \end{aligned}$$

and further noticing that

$$I_0 = \int_0^\infty e^{-x} \, \mathrm{d}x = \left[-e^{-x} \right]_0^\infty = 1,$$

we can see

$$I_t = t!,$$

and hence

$$\begin{aligned} &\int_0^\infty \frac{(1 - e^{-ax}) e^{-x}}{x} \, \mathrm{d}x \\ &= \sum_{t=1}^\infty \frac{(-1)^{t-1} a^t}{t!} \int_0^\infty x^{t-1} e^{-x} \, \mathrm{d}x \\ &= \sum_{t=1}^\infty \frac{(-1)^{t-1} a^t}{t!} (t-1)! \\ &= \sum_{t=1}^\infty \frac{(-1)^{t-1} a^t}{t} \\ &= -\sum_{t=1}^\infty \frac{(-a)^t}{t} \\ &= \ln(1+a), \end{aligned}$$

precisely as desired.

3. Using a substitution $x = e^{-u}$, when $x = 1$, $u = 0$, and when $x \to 0^+$, $u \to \infty$. Also,

$$\frac{\mathrm{d}x}{\mathrm{d}u} = -e^{-u},$$

and hence

$$\begin{aligned} &\int_0^1 \frac{x^p - x^q}{\ln x} \, \mathrm{d}x \\ &= \int_\infty^0 \frac{e^{-up} - e^{-uq}}{\ln e^{-u}} \cdot (-e^{-u}) \, \mathrm{d}u \\ &= \int_\infty^0 \frac{(e^{-up} - e^{-uq}) \, e^{-u}}{u} \, \mathrm{d}u \\ &= \int_0^\infty \frac{[(1 - e^{-up}) + (1 - e^{-uq})] \, e^{-u}}{u} \, \mathrm{d}u \\ &= \int_0^\infty \frac{(1 - e^{-up}) \, e^{-u}}{u} \, \mathrm{d}u - \int_0^\infty \frac{(1 - e^{-uq}) \, e^{-u}}{u} \, \mathrm{d}u \\ &= \ln(1+p) - \ln(1+q). \end{aligned}$$