2018.2 Question 4

1. By the identity, we have

$$\cos x + \cos 4x = 2\cos\frac{5}{2}x\cos\frac{3}{2}x,$$

and

$$\cos 2x + \cos 3x = 2\cos\frac{5}{2}x\cos\frac{1}{2}x.$$

Hence, we have

$$\cos x + 3\cos 2x + 3\cos 3x = 2\cos\frac{5}{2}x\left(\cos\frac{3}{2}x + 3\cos\frac{1}{2}x\right) = 0.$$

Hence, either

$$\cos\frac{5}{2}x = 0,$$

or

$$\cos\frac{3}{2}x + 3\cos\frac{1}{2}x = 0.$$

In the first case, we have $\frac{5}{2}x = \frac{1}{2}\pi + k\pi$ for $k \in \mathbb{Z}$, and hence

$$x = \frac{1+2k}{5}\cdot\pi.$$

Since $0 \leq x \leq 2\pi$, we have

$$0 \leq \frac{1+2k}{5} \leq 2,$$

and hence

$$0 \leq 1 + 2k \leq 10,$$

giving $k = 0, 1, 2, 3, 4$. Hence, the solutions are

$$x = \frac{1}{5}\pi, x = \frac{3}{5}\pi, x = \pi, x = \frac{7}{5}\pi, x = \frac{9}{5}\pi.$$

In the second case, notice that

$$\begin{aligned}\cos 3t &= \cos(2t + t)\\ &= \cos 2t\cos t - \sin 2t\sin t\\ &= (\cos^2 t - \sin^2 t)\cos t - 2\sin^2 t\cos t\\ &= \cos^3 t - 3\sin^2\cos t.\end{aligned}$$

Hence,

$$\cos\frac{3}{2}x + 3\cos\frac{1}{2}x = 0 \iff \cos^3\frac{1}{2}x - 3\sin^2\frac{1}{2}x\cos\frac{1}{2}x + 3\cos\frac{1}{2}x = 0,$$

and using the identity $\sin^2 t + \cos^2 t = 1$, this simplifies to

$$\cos^3\frac{1}{2}x + 3\cos^3\frac{1}{2}x = 0,$$

which is

$$\cos\frac{1}{2}x = 0.$$

This gives

$$\frac{1}{2}x = \frac{\pi}{2} + k\pi$$

for $k \in \mathbb{Z}$, and hence

$$x = (1 + 2k)\pi.$$

Since $0 \leq x \leq 2\pi$, the only k valid is $k = 0$, and this solves to $x = \pi$.

Hence, all the solutions to this equation is

$$x \in \left\{\frac{1}{5}\pi, \frac{3}{5}\pi, \pi, \frac{7}{5}\pi, \frac{9}{5}\pi\right\}.$$

2. Using the given identity, we have

$$\cos(x+y)+\cos(x-y)=2\cos x\cos y.$$

Hence, the original equation simplifies to

$$2\cos x\cos y-\cos 2x=1.$$

Using the identity $\cos 2x=2\cos^2 x-1$, and this gives

$$2\cos x\cos y-(2\cos^2 x-1)=1,$$

and hence

$$2\cos x\cos y-2\cos^2 x=0,$$

which means

$$\cos x(\cos y-\cos x)=0,$$

and hence $\cos x=0$ or $\cos y-\cos x=0$.

The first one gives us $x=\frac{\pi}{2}$ in the range $x\in[0,\pi]$.

Since cos is one-to-one when restricted to $[0,\pi]$, the second one is equivalent to $\cos y=\cos x$ which is equivalent to $x=y$.

The specific value is $x=\frac{\pi}{2}$.

3. Using the identity given, we have

$$\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2},$$

and

$$\cos(x+y)=2\cos^2\frac{x+y}{2}-1.$$

Let $u=\frac{x+y}{2}$ and $v=\frac{x-y}{2}$. We have $0\le u\le\pi$ and $-\frac{\pi}{2}\le v\le\frac{\pi}{2}$, and the original equation simplifies to

$$2\cos u\cos v-2\cos^2 u+1=\frac{3}{2},$$

and hence

$$4\cos u\cos v-4\cos^2 u+2=3,$$

and

$$4\cos^2 u-4\cos u\cos v+1=0.$$

Since $1=\cos^2 v+\sin^2 v$, we have

$$4\cos^2 u-4\cos u\cos v+\cos^2 v=-\sin^2 v,$$

and hence

$$(2\cos u-\cos v)^2=-\sin^2 v.$$

The left-hand side is non-negative, and the right-hand side is non-positive. Hence, the only way for the equal sign to take place is when both sides are zero, which is

$$2\cos u=\cos v,\sin v=0.$$

Within this range of v, the only case where $\sin v=0$ is when $v=0$, and hence $2\cos u=1$, $\cos u=\frac{1}{2}$, leading to $u=\frac{\pi}{3}$.

Hence, $x=u+v=\frac{\pi}{3}$, and $y=u-v=\frac{\pi}{3}$, and the only solution is

$$(x,y)=\left(\frac{\pi}{3},\frac{\pi}{3}\right).$$