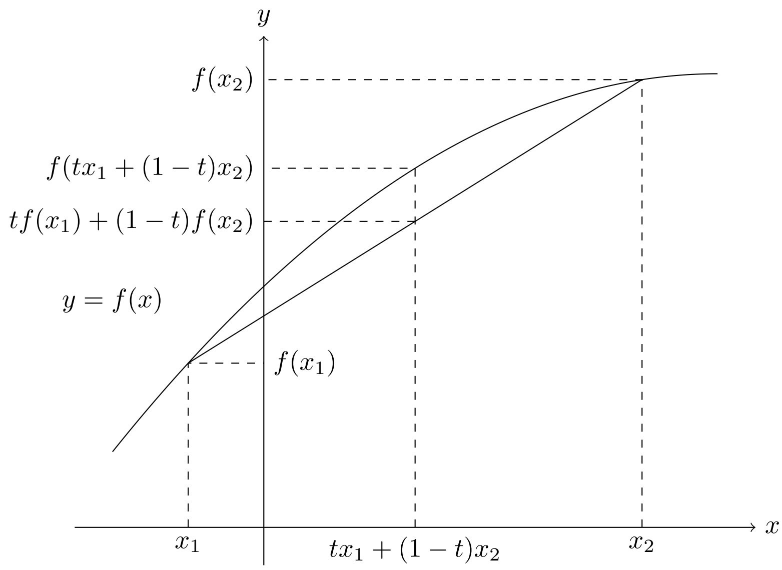
2018.2 Question 2



If $f''(x) < 0$, this means $f'(x)$ is decreasing, i.e. the gradient of a tangent to the curve $y = f(x)$ is decreasing. Assume, B.W.O.C., that some $f(x)$ satisfies this condition but is not convex. This means that there exists some $a < x_1 < x_2 < b$ and some $0 < t < 1$ that

$$tf(x_1) + (1-t)f(x_2) \geq f(tx_1 + (1-t)x_2).$$

This means that some point on the chord connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is above the graph of the function at that point with x-coordinate $tx_1 + (1-t)x_2$. Hence, the gradient of that function must be less than the gradient of the chord at that point, and since $f''(x) < 0$, the function must continue to have a gradient of less than this, and hence cannot pass through $(x_2, f(x_2))$.

Hence, this triple of (x_1, x_2, t) does not exist, and the function f must be concave on (a, b).

1. Let $x_1 = \frac{2u+v}{3}$ and $x_2 = \frac{v+2w}{3}$, and let $t = \frac{1}{2}$. We can see that $a < x_1, x_2 < b$ and hence we have

$$\frac{1}{2}f(x_1) + \frac{1}{2}f(x_2) \leq f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right),$$

which gives

$$\frac{1}{2}f\left(\frac{2u+v}{3}\right) + \frac{1}{2}f\left(\frac{v+2w}{3}\right) \leq f\left(\frac{u+v+w}{3}\right).$$

Let $x_1 = u$ and $x_2 = v$, and let $t = \frac{2}{3}$. We have

$$\frac{2}{3}f(u) + \frac{1}{3}f(v) \leq f\left(\frac{2u+v}{3}\right),$$

and let $x_1 = w$, $x_2 = v$, and let $t = \frac{2}{3}$, we have

$$\frac{2}{3}f(w) + \frac{1}{3}f(v) \leq f\left(\frac{2w+v}{3}\right).$$

Hence,

$$\begin{aligned}
f\left(\frac{u+v+w}{3}\right) &\geq \frac{1}{2}f\left(\frac{2u+v}{3}\right) + \frac{1}{2}f\left(\frac{v+2w}{3}\right) \\
&\geq \frac{1}{2}\cdot\left[\frac{2}{3}f(u) + \frac{1}{3}f(v)\right] + \frac{1}{2}\cdot\left[\frac{2}{3}f(w) + \frac{1}{3}f(v)\right] \\
&= \frac{1}{3}\left[f(u) + f(v) + f(w)\right],
\end{aligned}$$

which shows exactly what is desired.

2. Let $a = 0$ and $b = \pi$, and let $f(x) = \sin x$. We aim to show that f is concave, and notice that
$$f''(x) = -\sin x < 0$$
for all $0 < x < \pi$, so it is concave on $(0, \pi)$.
Angles in a triangle lie within $(0, \pi)$, and they must sum up to π. Hence, by applying the previous part, we have
$$\sin A + \sin B + \sin C \leq 3 \sin\left(\frac{A + B + C}{3}\right) = 3 \sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2},$$
as desired.

3. We keep $a = 0$ and $b = \pi$, and let $f(x) = \ln \sin x$. Note that
$$f'(x) = \frac{\cos x}{\sin x} = \cot x,$$
and hence
$$f''(x) = -\csc^2 x < 0$$
which shows that f is concave on $(0, \pi)$.
Hence,
$$\begin{aligned}\ln(\sin A \sin B \sin C) &= \ln \sin A + \ln \sin B + \ln \sin C \\ &\leq 3 \ln \sin\left(\frac{A + B + C}{3}\right) \\ &= 3 \ln \sin\left(\frac{\pi}{3}\right) \\ &= 3 \ln \frac{\sqrt{3}}{2} \\ &= \ln \frac{3\sqrt{3}}{8}.\end{aligned}$$
Since ln is a strictly increasing function, we can then conclude that
$$\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8},$$
as desired.