2018.2 Question 13

This setup gives a Markov Chain. Let the column vector $\mathbf{x}_n$ represent a state

$$\mathbf{x}_n = \begin{pmatrix} A_n \\ B_n \\ C_n \\ D_n \end{pmatrix},$$

and hence we have the components of the column vector must sum to 1. The initial state is defined by

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and the state transition matrix is

$$\mathbf{M} = \begin{pmatrix} 1/2 & 1/4 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 & 0 \\ 0 & 1/4 & 1/2 & 1/4 \\ 1/4 & 0 & 1/4 & 1/2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix},$$

which gives

$$\mathbf{x}_{n+1} = \mathbf{M}\mathbf{x}_n.$$

1. Notice that

$$\mathbf{x}_1 = \mathbf{M}\mathbf{x}_0 = \frac{1}{4} \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

   and hence $A_1 = \frac{1}{2}, B_1 = \frac{1}{4}, C_1 = 0, D_1 = \frac{1}{4}$.
   Also,

$$\mathbf{x}_2 = \mathbf{M}\mathbf{x}_1 = \frac{1}{4} \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \cdot \frac{1}{4} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 6 \\ 4 \\ 2 \\ 4 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix},$$

   and hence $A_2 = \frac{3}{8}, B_2 = \frac{1}{4}, C_2 = \frac{1}{8}, D_2 = \frac{1}{4}$.

2. We claim that $B_n = D_n$ for all n by symmetry, and notice that

$$B_{n+1} = \frac{1}{4} \cdot (A_n + 2B_n + C_n) = \frac{1}{4} \cdot (A_n + B_n + C_n + D_n) = \frac{1}{4},$$

   and

$$D_{n+1} = \frac{1}{4} \cdot (A_n + C_n + 2D_n) = \frac{1}{4} \cdot (A_n + B_n + C_n + D_n) = \frac{1}{4},$$

   so that $B_n = D_n = \frac{1}{4}$ for all $n \geq 1$. (For $n = 0$, $B_n = D_n = 0$).
   Hence, for $n \geq 1$, we have

$$A_{n+1} = \frac{1}{4}(2A_n + B_n + D_n) = \frac{1}{4}\left(2A_n + \frac{1}{2}\right) = \frac{1}{2}A_n + \frac{1}{8},$$

   which means

$$A_{n+1} - \frac{1}{4} = \frac{1}{2}\left(A_n - \frac{1}{4}\right),$$

   which shows that $A_n - \frac{1}{4}$ is a geometric sequence with common ratio $\frac{1}{2}$. The initial term of the geometric sequence is $A_1 - \frac{1}{4} = \frac{1}{4}$, and hence

$$A_n - \frac{1}{4} = \frac{1}{2^{n+1}},$$

which shows $A_n = \frac{1}{4} + \frac{1}{2^{n+1}}$ for $n \geq 1$.

Also, C_n has the same inductive relationship as A_n, the only difference being that the initial term is $C_1 - \frac{1}{4} = -\frac{1}{4}$, and hence

$$C_n - \frac{1}{4} = -\frac{1}{2^{n+1}},$$

which shows $C_n = \frac{1}{4} - \frac{1}{2^{n+1}}$ for $n \geq 1$.

Hence, we have

$$\mathbf{x}_n = \begin{pmatrix} A_n \\ B_n \\ C_n \\ D_n \end{pmatrix} = \begin{cases} (1, 0, 0, 0)^\intercal, & n = 0, \\ \left(\frac{1}{4} + \frac{1}{2^{n+1}}, \frac{1}{4}, \frac{1}{4} - \frac{1}{2^{n+1}}, \frac{1}{4}\right)^\intercal, & \text{otherwise.} \end{cases}$$