## 2018.2 Question 12

1. If $h$ consecutive heads are thrown, then the person will earn £$h$, and the probability of this happening is $p^h$.

   If this did not happen, then the game must have already ended before reaching $h$ heads (since there must be a tail), and the person will earn nothing.

   Hence, the expected earning is $E(h) = hp^h$, which gives

   $$E(h) = \frac{hN^h}{(N+1)^h}.$$

   Notice that
   $$\frac{E(h+1)}{E(h)} = \frac{(h+1)N^{h+1}/(N+1)^{h+1}}{hN^h/(N+1)^h} = \frac{(h+1)N}{h(N+1)}.$$

   We have
   $$\frac{E(h+1)}{E(h)} - 1 = \frac{(hN+N)-(hN+h)}{hN+h} = \frac{N-h}{hN+h},$$

   which shows that $E(h+1) > E(h)$ when $h < N$, and $E(h+1) < E(h)$ when $h > N$, and $E(h+1) = E(h)$ when $h = N$.

   This means that $E(h)$ will increase until $h = N$, where $E(N) = E(N+1)$, and decrease after $h = N+1$.

   This means the expected earnings can be maximised when $h = N$ or $h = N+1$, which shows when $h = N$, the earnings is maximised.

2. There are two cases: either the person earns £$h$ (when there are $h$ heads thrown before the game ends) with some probability (that we would like to find), or the game ends before there are $h$ heads thrown.

   To find the probability in the first case, let there be $t$ cases where a tail appears, and there must be $h$ cases where a head appears. The final throw must be a head, and the tail must appear singularly (which means any two consecutive tails must have a head in between), which shows that $0 \le t \le h$.

   There are $h-1$ heads that are free to 'move', and $t$ tails have $t-1$ gaps in between, which takes away at least $t-1$ heads to separate them. The rest of the $h-t$ heads are free to be within any of the $t+1$ spaces that are separated by the $t$ tails, which is equivalent of choosing $t$ to be heads from a total $(h-t)+t = h$ remaining throws.

   Therefore, for each $t$, the number of arrangements there are is

   $$\binom{h}{t},$$

   and the probability of this happening is

   $$p^h \cdot (1-p)^t.$$

   Therefore, the probability desired is

   $$\sum_{t=0}^{h} \binom{h}{t} p^h (1-p)^t = p^h \sum_{t=0}^{h} \binom{h}{t} 1^{h-t}(1-p)^t = p^h(1+1-p)^h = p^h(2-p)^h,$$

   and the expected earnings in terms of $h$ is

   $$E(h) = hp^h(2-p)^h = h\left(\frac{N}{N+1}\right)^h \left(\frac{N+2}{N+1}\right)^h = \frac{hN^h(N+2)^h}{(N+1)^{2h}}$$

   as desired.

   When $N = 2$,
   $$E(h) = \frac{h2^h4^h}{3^{2h}} = \frac{h2^{3h}}{3^{2h}}.$$

Notice that

$$\frac{E(h+1)}{E(h)} = \frac{(h+1)2^{3h+3}/3^{2h+2}}{h2^{3h}/3^{2h}} = \frac{8(h+1)}{9h},$$

and hence

$$\frac{E(h+1)}{E(h)} - 1 = \frac{8-h}{9h},$$

which shows that $E(h+1) > E(h)$ when $h < 8$, and $E(h+1) < E(h)$ when $h > 8$, and $E(h+1) = E(h)$ when $h = 8$.

This shows that $E(8) = E(9)$ gives the maximum expected winnings, which is given by

$$\frac{8 \cdot 2^{24}}{3^{16}} = \frac{2^{27}}{3^{16}}.$$

Since $\log_3 2 \approx 0.63$, we have $2 \approx 3^{0.63}$, and hence

$$\frac{2^{27}}{3^{16}} \approx \frac{3^{27 \cdot 0.63}}{3^{16}} = 3^{27 \cdot 0.63 - 16} = 3^{1.01} \approx 3,$$

and this shows that the maximum value of expected winnings is approximately £3.