2018.2 Question 1

First, notice that $x = 0$ must not be a root to this quartic equation. Therefore, we can divide both sides by x^2, and the original equation is equivalent to

$$x^2 + \frac{1}{x^2} + a\left(x + \frac{1}{x}\right) + b = 0,$$

and this rearranges to

$$\left(x + \frac{1}{x}\right)^2 + a\left(x + \frac{1}{x}\right) + (b - 2) = 0.$$

Notice that

$$k + \frac{1}{k} = \frac{1}{k^{-1}} + k^{-1} = k^{-1} + \frac{1}{k^{-1}},$$

so if $x = k$ satisfies this equation, then $x = k^{-1}$ also satisfies this equation.

Notice that the range of $t = x + \frac{1}{x}$ for non-zero real x is $t \in (-\infty, -2] \cup [2, \infty)$.

Since it is given that all the roots are real, it must be the case that the quadratic equation

$$t^2 + at + (b - 2) = 0$$

produces two real roots situated within $(-\infty, -2] \cup [2, \infty)$.

Notice that for $t \in (-\infty, -2] \cup [2, \infty)$, the equation

$$x + \frac{1}{x} = t$$

has precisely two real roots for $t \neq \pm 2$, and precisely one $x = \pm 1$ for $t = \pm 2$.

1. In this case, by the previous analysis, the only possibility is that $x_1 = x_2 = x_3 = x_4 = \mp 1$. This means that
$$x^4 + ax^3 + bx^2 + ax + 1 = (x \pm 1)^4 = x^4 \pm 4x^3 + 6x^2 \pm 4x + 1,$$
and hence $(a, b) = (\pm 4, 6)$.

2. Since there are exactly three distinct roots for x, this means that the one which repeated must be $x_1 = x_2 = \pm 1$, which leads to $t_1 = \pm 2$, and those two which does not leads to $t_2 \neq \pm 2$.

   Putting $t_1 = \pm 2$ into the quadratic equation in t, we have
$$4 \pm 2a + (b - 2) = 0,$$
and hence
$$b = \mp 2a - 2,$$
precisely as desired.

3. When $b = 2a - 2$, we have
$$t^2 + at + (2a - 4) = 0,$$
which solves to $t_1 = -2$, $t_2 = -a + 2$.

   For $x + \frac{1}{x} = t_1 = -2$, this solves to $x_1 = x_2 = -1$.

   For $x + \frac{1}{x} = t_2 = -a + 2$, this rearranges to
$$x^2 + (a - 2)x + 1 = 0,$$
and hence the two roots are
$$x_{3,4} = \frac{-(a-2) \pm \sqrt{(a-2)^2 - 4}}{2} = \frac{-a + 2 \pm \sqrt{a^2 - 4a}}{2}$$

4. We first look at necessary condition. Given the equation has precisely two roots, we have $b = \pm 2a - 2$, and hence the quadratic equation in t becomes

$$t^2 + at + (\pm 2a - 4) = 0.$$

$t_1 = \mp 2$ must be a root, and notice that this factorises to

$$t^2 + at + (\pm 2a - 4) = (t \pm 2)(t - (-a \pm 2)),$$

and hence the other root is $t_2 = -a \pm 2$.

As discussed before, we must have that $t_2 < -2$ or $t_2 > 2$ to produce two distinct roots for x, and hence

$$-a \pm 2 < -2 \text{ or } -a \pm 2 > 2,$$

and hence

$$a \mp 2 > 2 \text{ or } a \mp 2 < -2,$$

and hence

$$a > 2 \pm 2 \text{ or } a < -2 \pm 2.$$

Therefore, a necessary condition is $b = \pm 2a - 2$, and $a \in (-\infty, -2 \pm 2) \cup (2 \pm 2, \infty)$.

We would like to show that this is a sufficient condition as well. If $b = \pm 2a - 2$ and $a \in (-\infty, -2 \pm 2) \cup (2 \pm 2, \infty)$, we have the quadratic in t simplifies to

$$t^2 + at + (\pm 2a - 4) = (t \pm 2)(t - (-a \pm 2)) = 0.$$

This gives roots $t_1 = \mp 2$ which in turn gives $x_1 = x_2 = \mp 1$, and $t_2 = -a \pm 2$. In the second case, since

$$a \in (-\infty, -2 \pm 2) \cup (2 \pm 2, \infty),$$

we must have

$$a \mp 2 \in (-\infty, -2) \cup (2, \infty)$$

and hence

$$-a \pm 2 \in (-\infty, -2) \cup (2, \infty).$$

This shows that there are two distinct xs corresponding to t_2, both of which are not equal to ± 1.

Hence, in this case, the original equation has 3 distinct roots precisely, and

$$b = \pm 2a - 2, a \in (-\infty, -2 \pm 2) \cup (2 \pm 2, \infty)$$

is a necessary and sufficient condition for the original equation to have precisely 3 distinct real roots.

The following is to simplify this to what is written in the mark scheme. $b = \pm 2a - 2$ is equivalent to $b + 2 = \pm 2a$, and $(b+2)^2 = 4a^2$.

The second part is equivalent to $a \mp 2 \in (-\infty, -2) \cup (2, \infty)$, i.e.

$$(a \mp 2)^2 = a^2 \mp 4a + 4 > 4,$$

i.e.

$$a^2 > \pm 4a = 2 \pm 2a = 2(b+2) = 2b + 4,$$

precisely what is in the mark scheme.