2017.3 Question 8

We have

$$\begin{aligned}\sum_{m=1}^{n} a_m(b_{m+1}-b_m) &= \sum_{m=1}^{n} a_m b_{m+1} - \sum_{m=1}^{n} a_m b_m \\ &= -\sum_{m=0}^{n-1} b_{m+1}a_{m+1} + \sum_{m=1}^{n} b_{m+1}a_m \\ &= -\sum_{m=1}^{n} b_{m+1}a_{m+1} + \sum_{m=1}^{n} b_{m+1}a_m + a_{n+1}b_{n+1} - a_1 b_1 \\ &= a_{n+1}b_{n+1} - a_1 b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1}-a_m),\end{aligned}$$

as desired.

1. Let $a_m = 1$. On one hand, we have

$$\begin{aligned}\sum_{m=1}^{n} a_m(b_{m+1}-b_m) &= \sum_{m=1}^{n} [\sin(m+1)x - \sin mx] \\ &= \sum_{m=1}^{n} 2\cos\left(\frac{(m+1)x+mx}{2}\right)\sin\left(\frac{(m+1)x-mx}{2}\right) \\ &= 2\sum_{m=1}^{n} \cos\left(m+\frac{1}{2}\right)x \sin\frac{x}{2} \\ &= 2\sin\frac{x}{2}\sum_{m=1}^{n}\cos\left(m+\frac{1}{2}\right)x.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sum_{m=1}^{n} a_m(b_{m+1}-b_m) &= a_{n+1}b_{n+1} - a_1 b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1}-a_m) \\ &= \sin(n+1)x - \sin x.\end{aligned}$$

Therefore, by rearranging, we have

$$\sum_{m=1}^{n}\cos\left(m+\frac{1}{2}\right)x = \frac{1}{2}\left[\sin(n+1)x - \sin x\right]\operatorname{cosec}\frac{1}{2}x$$

as desired.

2. Let $a_m = m$, and let $b_m = \cos\left(m-\frac{1}{2}\right)x$. We have the identity

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).$$

Therefore, we have

$$\begin{aligned}\sum_{m=1}^{n} a_m(b_{m+1}-b_m) &= \sum_{m=1}^{n} m\cdot\left[\cos\left(m+\frac{1}{2}\right)x - \cos\left(m-\frac{1}{2}\right)x\right] \\ &= \sum_{m=1}^{n} -2m\sin mx \sin\frac{1}{2}x \\ &= -2\sin\frac{1}{2}x\sum_{m=1}^{n} m\sin mx,\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m=1}^{n} a_m(b_{m+1} - b_m) \\
&= a_{n+1}b_{n+1} - a_1b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1} - a_m) \\
&= (n+1)\cos\left(n + \frac{1}{2}\right)x - 1 \cdot \cos\frac{1}{2}x - \sum_{m=1}^{n} \cos\left(m + \frac{1}{2}\right)x \cdot 1 \\
&= (n+1)\cos\left(n + \frac{1}{2}\right)x - \cos\frac{1}{2}x - \sum_{m=1}^{n} \cos\left(m + \frac{1}{2}\right)x \\
&= (n+1)\cos\left(n + \frac{1}{2}\right)x - \cos\frac{1}{2}x - \frac{1}{2}\left(\sin(n+1)x - \sin x\right)\operatorname{cosec}\frac{1}{2}x \\
&= \frac{1}{2}\operatorname{cosec}\frac{1}{2}x\left[2(n+1)\cos\left(n + \frac{1}{2}\right)x\sin\frac{1}{2}x - 2\cos\frac{1}{2}x\sin\frac{1}{2}x - \left(\sin(n+1)x - \sin x\right)\right] \\
&= \frac{1}{2}\operatorname{cosec}\frac{1}{2}x\left[(n+1)\left(\sin(n+1)x - \sin nx\right) - (\sin x - \sin 0) - \left(\sin(n+1)x - \sin x\right)\right] \\
&= \frac{1}{2}\operatorname{cosec}\frac{1}{2}x\left[n\sin(n+1)x - (n+1)\sin nx\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
-2\sin\frac{1}{2}x\sum_{m=1}^{n} m\sin mx &= \frac{1}{2}\operatorname{cosec}\frac{1}{2}x\left[n\sin(n+1)x - (n+1)\sin nx\right] \\
\sum_{m=1}^{n} m\sin mx &= -\frac{1}{4}\operatorname{cosec}^2\frac{1}{2}x\left[n\sin(n+1)x - (n+1)\sin nx\right],
\end{aligned}$$

and therefore, $p = -\frac{1}{4}n$, $q = \frac{1}{4}(n+1)$.