2017.3 Question 7

$$\begin{aligned}
\frac{x^2}{a^2}+\frac{y^2}{b^2} &= \left(\frac{1-t^2}{1+t^2}\right)^2+\left(\frac{2t}{1+t^2}\right)^2 \\
&= \frac{\left(1-t^2\right)^2+(2t)^2}{\left(1+t^2\right)^2} \\
&= \frac{1-2t^2+t^4+4t^2}{\left(1+t^2\right)^2} \\
&= \frac{1+2t^2+t^4}{\left(1+t^2\right)^2} \\
&= \frac{\left(1+t^2\right)^2}{\left(1+t^2\right)^2} \\
&= 1
\end{aligned}$$

as desired, so T lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

1. The gradient of L must satisfy that

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t} \\
&= \frac{b}{a}\cdot\frac{\mathrm{d}\left(2t/(1+t^2)\right)/\,\mathrm{d}t}{\mathrm{d}\left((1-t^2)/(1+t^2)\right)/\,\mathrm{d}t} \\
&= \frac{b}{a}\cdot\frac{2\cdot(1+t^2)-2t\cdot 2t}{-2t\cdot(1+t^2)-(1-t^2)\cdot 2t} \\
&= \frac{b}{a}\cdot\frac{2+2t^2-4t^2}{-2t-2t^3-2t+2t^3} \\
&= \frac{b}{a}\cdot\frac{1-t^2}{-2t}.
\end{aligned}$$

Therefore, we have a general point $(X,Y)\in L$ satisfy that

$$\begin{aligned}
Y-\frac{2bt}{1+t^2} &= \frac{b}{a}\cdot\frac{1-t^2}{-2t}\cdot\left(X-\frac{a(1-t^2)}{1+t^2}\right) \\
(1+t^2)Y-2bt &= \frac{b}{a}\cdot\frac{1-t^2}{-2t}\cdot\left((1+t^2)X-a(1-t^2)\right) \\
(-2at)(1+t^2)Y-(-2at)(2bt) &= b\cdot(1-t^2)\cdot\left((1+t^2)X-a(1-t^2)\right) \\
(-2at)(1+t^2)Y &= b(1-t^2)(1+t^2)X-ab(1-t^2)^2-4abt^2 \\
(-2at)(1+t^2)Y &= b(1-t^2)(1+t^2)X-ab(1+t^2)^2 \\
-2atY &= b(1-t^2)X-ab(1+t^2) \\
ab(1+t^2)-2atY-b(1-t^2)X &= 0 \\
(a+X)bt^2-2aYt+b(a-X) &= 0
\end{aligned}$$

as desired.

Now if we fix X,Y and solve for t, there are two solutions to this quadratic equation exactly when

$$\begin{aligned}
(2aY)^2-4(a+X)b\cdot b(a-X) &> 0 \\
(aY)^2-(a+X)(a-X)b^2 &> 0 \\
a^2Y^2 &> (a^2-X^2)b^2,
\end{aligned}$$

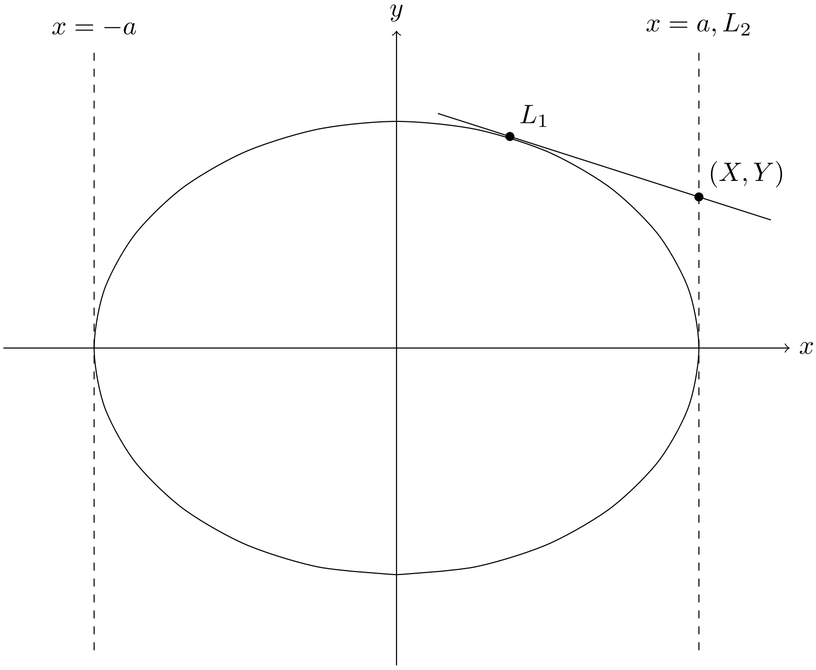
which corresponds to two distinct points on the ellipse.

Since $a^2Y^2 > (a^2 - X^2)b^2$, we have $\frac{Y^2}{b^2} > 1 - \frac{X^2}{a^2}$ by dividing through a^2b^2 on both sides, i.e.

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} > 1,$$

which means when the point (X, Y) lies outside the ellipse.

This also holds when $X^2 = a^2$, i.e. when the point (X, Y) lies on the pair of lines $X = \pm A$. Here, the condition is simply $a^2Y^2 > 0$, which gives $Y \neq 0$. One of the tangents will be the vertical line $X = \pm A$ (whichever one the point lies on), and the other one as a non-vertical (as shown when $X = a$, the tangents being L_1 and L_2).



2. By Vieta's Theorem, we have

$$pq = \frac{b(a - X)}{b(a + X)} \implies (a + X)pq = a - X,$$

as desired, and

$$p + q = -\frac{-2aY}{(a + X)b} = \frac{2aY}{(a + X)b}.$$

Let $X = 0$ for the equation in L,

$$\begin{aligned} abt^2 - 2aYt + ba &= 0 \\ bt^2 - 2Yt + b &= 0 \\ Y &= \frac{b(1 + t^2)}{2t}. \end{aligned}$$

Therefore,

$$\begin{aligned} y_1 + y_2 &= \frac{b(1 + p^2)}{2p} + \frac{b(1 + q^2)}{2q} \\ &= \frac{b\left[(1 + p^2)q + (1 + q^2)p\right]}{2pq} \\ &= 2b, \end{aligned}$$

therefore we have

$$4pq = (1 + p^2)q + (1 + q^2)p = (p + q)(1 + pq).$$

Therefore,

$$
\begin{aligned}
4 \cdot \frac{a - X}{a + X} &= \frac{2aY}{(a + X)b} \cdot \frac{2a}{a + X} \\
a - X &= \frac{a^2 Y}{b(a + X)} \\
(a - X)(a + X)b &= a^2 Y \\
(a^2 - X^2)b &= a^2 Y \\
1 - \frac{X^2}{a^2} &= \frac{Y}{b} \\
\frac{X^2}{a^2} + \frac{Y}{b} &= 1,
\end{aligned}
$$

as desired.