2017.3 Question 6

1. Consider the substitution $u = \frac{1}{v}$.

   When $u \to 0^+$, $v \to +\infty$.

   When $u = x$, $v = \frac{1}{x}$.

   We also have

   $$\mathrm{d}u = -\frac{1}{v^2}\,\mathrm{d}v.$$

   Therefore,

   $$\begin{aligned} T(x) &= \int_0^x \frac{\mathrm{d}u}{1+u^2} \\ &= \int_{+\infty}^{\frac{1}{x}} -\frac{1}{v^2} \cdot \frac{1}{1+\frac{1}{v^2}}\,\mathrm{d}v \\ &= \int_{\frac{1}{x}}^{+\infty} \frac{\mathrm{d}v}{1+v^2} \\ &= \int_0^{+\infty} \frac{\mathrm{d}v}{1+v^2} - \int_0^{\frac{1}{x}} \frac{\mathrm{d}v}{1+v^2} \\ &= T_\infty - T(x^{-1}), \end{aligned}$$

   as desired.

2. When $u \neq a^{-1}$, we have

   $$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}u} &= \frac{\mathrm{d}}{\mathrm{d}u}\frac{u+a}{1-au} \\ &= \frac{1\cdot(1-au) + a\cdot(u+a)}{(1-au)^2} \\ &= \frac{1-au+au+a^2}{(1-au)^2} \\ &= \frac{1+a^2}{(1-au)^2}. \end{aligned}$$

   Also, notice that

   $$\begin{aligned} \frac{1+v^2}{1+u^2} &= \frac{1+\left(\frac{u+a}{1-au}\right)^2}{1+u^2} \\ &= \frac{(1-au)^2+(u+a)^2}{(1+u^2)(1-au)^2} \\ &= \frac{1-2au+a^2u^2+u^2+2au+a^2}{(1+u^2)(1-au)^2} \\ &= \frac{(1+a^2)(1+u^2)}{(1-au)^2(1+u^2)} \\ &= \frac{1+a^2}{(1-au)^2}. \end{aligned}$$

   Therefore, $\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{1+v^2}{1+u^2}$ as desired.

Consider the substitution $v = \frac{u+a}{1-au}$. When $u = 0$, $v = a$. When $u = x, v = \frac{x+a}{1-ax}$. Therefore,

$$\begin{aligned}
T(x) &= \int_0^x \frac{\mathrm{d}u}{1+u^2} \\
&= \int_a^{\frac{x+a}{1-ax}} \frac{1+u^2}{1+v^2} \cdot \frac{\mathrm{d}v}{1+u^2} \\
&= \int_a^{\frac{x+a}{1-ax}} \frac{\mathrm{d}v}{1+v^2} \\
&= \int_0^{\frac{x+a}{1-ax}} \frac{\mathrm{d}v}{1+v^2} - \int_0^a \frac{\mathrm{d}v}{1+v^2} \\
&= T\left(\frac{x+a}{1-ax}\right) - T(a),
\end{aligned}$$

as desired.

If we substitute $T(x) = T_\infty - T(x^{-1})$ and $T(a) = T_\infty - T(a^{-1})$, we can see that

$$\begin{aligned}
T(x) &= T\left(\frac{x+a}{1-ax}\right) - T(a) \\
T_\infty - T(x^{-1}) &= T\left(\frac{x+a}{1-ax}\right) - \left[T_\infty - T(a^{-1})\right] \\
T(x^{-1}) &= 2T_\infty - T\left(\frac{x+a}{1-ax}\right) - T(a^{-1}),
\end{aligned}$$

as desired.

Now, let $y = x^{-1}$ and $b = a^{-1}$. Then

$$\begin{aligned}
\frac{x+a}{1-ax} &= \frac{y^{-1}+b^{-1}}{1-b^{-1}y^{-1}} \\
&= \frac{b+y}{by-1}.
\end{aligned}$$

This gives us

$$T(y) = 2T_\infty - T\left(\frac{b+y}{by-1}\right) - T(b),$$

as desired.

3. Let $y = b = \sqrt{3}$. We can easily verify that $b > 0$ and $y > \frac{1}{b}$. Therefore,

$$T(\sqrt{3}) = 2T_\infty - T\left(\frac{\sqrt{3}+\sqrt{3}}{3-1}\right) - T(\sqrt{3}),$$

which simplified, gives us $T(\sqrt{3}) = \frac{2}{3}T_\infty$ as desired.

In $T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$, let $x = a = \sqrt{2} - 1$, we can verify that $a > 0$ and $x < \frac{1}{a}$, therefore we have

$$\begin{aligned}
T(\sqrt{2}-1) &= T\left(\frac{(\sqrt{2}-1)+(\sqrt{2}-1)}{1-(\sqrt{2}-1)\cdot(\sqrt{2}-1)}\right) - T(\sqrt{2}-1), \\
T(\sqrt{2}-1) &= T\left(\frac{2\sqrt{2}-2}{1-\left(2+1-2\sqrt{2}\right)}\right) - T(\sqrt{2}-1), \\
T(\sqrt{2}-1) &= T\left(\frac{2\sqrt{2}-2}{2\sqrt{2}-2}\right) - T(\sqrt{2}-1), \\
2T(\sqrt{2}-1) &= T(1).
\end{aligned}$$

In $T(x) = T_\infty - T(x^{-1})$, let $x = 1$. We have

$$\begin{aligned} T(1) &= T_\infty - T(1), \\ 2T(1) &= T_\infty. \end{aligned}$$

Therefore, $T(\sqrt{2} - 1) = \frac{1}{4} T_\infty$, as desired.