2017.3 Question 5

Since we have $x = r\cos\theta$ and $y = r\sin\theta$, and $r = f(\theta)$, we have

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\cdot\cos\theta + r\cdot\frac{\mathrm{d}\cos\theta}{\mathrm{d}\theta}\\ &= f'(\theta)\cos\theta - f(\theta)\sin\theta,\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\cdot\sin\theta + r\cdot\frac{\mathrm{d}\sin\theta}{\mathrm{d}\theta}\\ &= f'(\theta)\sin\theta + f(\theta)\cos\theta,\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}}\\ &= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}\\ &= \frac{f'(\theta)\tan\theta + f(\theta)}{f'(\theta) - f(\theta)\tan\theta}.\end{aligned}$$

For the two curves, we must have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_f\cdot\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_g = -1$$

for them to meet at right angles. Therefore,

$$\begin{aligned}\frac{f'(\theta)\tan\theta + f(\theta)}{f'(\theta) - f(\theta)\tan\theta}\cdot\frac{g'(\theta)\tan\theta + g(\theta)}{g'(\theta) - g(\theta)\tan\theta} &= -1\\ (f'(\theta)\tan\theta + f(\theta))\cdot(g'(\theta)\tan\theta + g(\theta)) &= -(f'(\theta) - f(\theta)\tan\theta)\cdot(g'(\theta) - g(\theta)\tan\theta)\\ f'(\theta)g'(\theta)(1+\tan^2\theta) + f(\theta)g(\theta)(1+\tan^2\theta) &= 0\\ f'(\theta)g'(\theta) + f(\theta)g(\theta) &= 0.\end{aligned}$$

We have $f\left(-\frac{\pi}{2}\right) = 4$. Let

$$g_a(\theta) = a(1+\sin\theta).$$

Therefore,

$$g_a'(\theta) = a\cos\theta,$$

and we have

$$f'(\theta)(a\cos\theta) + f(\theta)a(1+\sin\theta) = 0,$$

and therefore

$$\frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta}\cos\theta = -f(\theta)(1+\sin\theta).$$

By separating variables we have

$$\frac{\mathrm{d}f(\theta)}{f(\theta)} = -\frac{\mathrm{d}\theta(1+\sin\theta)}{\cos\theta}.$$

Notice that

$$-\frac{1+\sin\theta}{\cos\theta} = -\frac{(1-\sin\theta)(1+\sin\theta)}{(1-\sin\theta)\cos\theta} = -\frac{\cos\theta}{1-\sin\theta} = \frac{\cos\theta}{\sin\theta - 1},$$

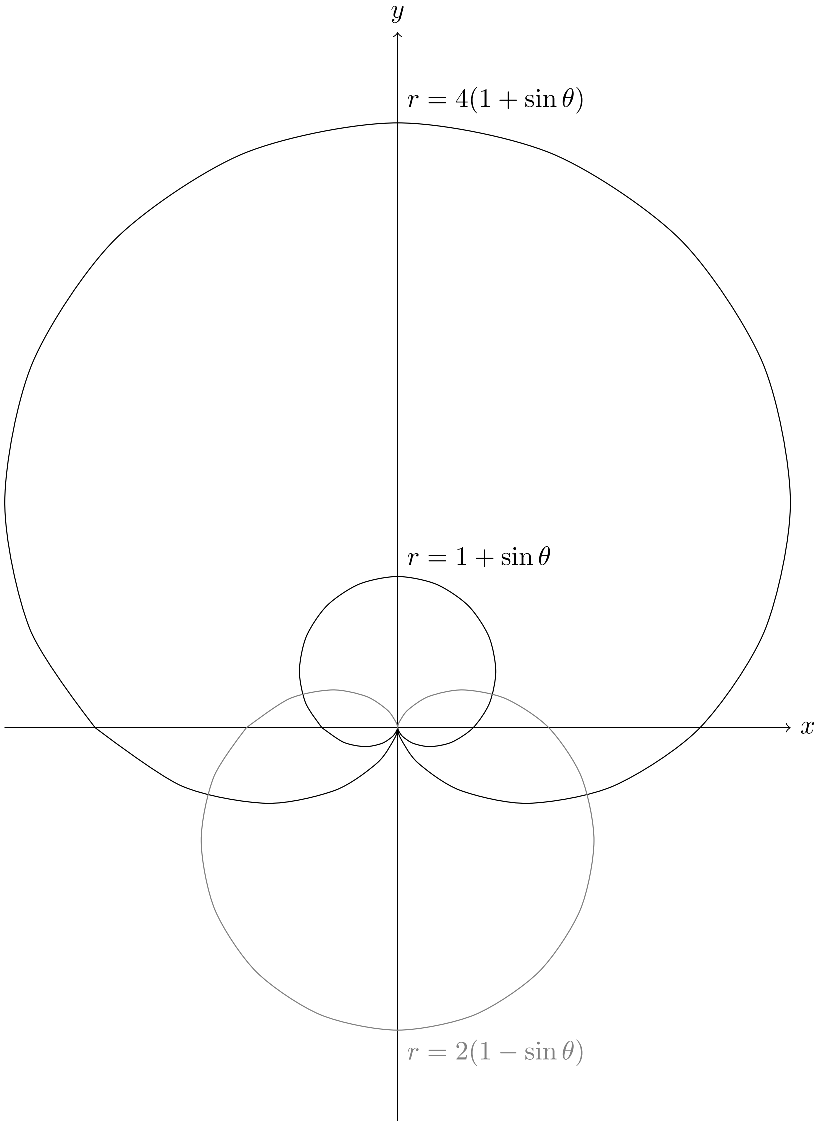
integrating both sides gives us

$$\ln f(\theta) = \ln|\sin\theta - 1| + C = \ln(1-\sin\theta) + C,$$

which gives

$$f(\theta) = A(1-\sin\theta).$$

Since $f\left(-\frac{\pi}{2}\right) = 4$, we have $2A = 4$ and $A = 2$, therefore $f(\theta) = 2(1-\sin\theta)$.


y
x
$r = 4(1 + \sin\theta)$
$r = 1 + \sin\theta$
$r = 2(1 - \sin\theta)$