2017.3 Question 4

1. Notice that $a = e^{\ln a}$ and hence $a^x = e^{x \ln a}$, $a^{\frac{x}{\ln a}} = e^x$we have

$$\begin{aligned} F(y) &= \exp\left(\frac{1}{y}\int_0^y \ln f(x)\,\mathrm{d}x\right) \\ &= a^{\frac{1}{y \ln a} \cdot \int_0^y \ln f(x)\,\mathrm{d}x} \\ &= a^{\frac{1}{y} \cdot \int_0^y \frac{\ln f(x)}{\ln a}\,\mathrm{d}x} \\ &= a^{\frac{1}{y} \cdot \int_0^y \log_a f(x)\,\mathrm{d}x} \end{aligned}$$

   as desired.

2. We have

$$\begin{aligned} H(y) &= \exp\left(\frac{1}{y}\int_0^y \ln f(x)g(x)\,\mathrm{d}x\right) \\ &= \exp\left[\frac{1}{y}\int_0^y (\ln f(x) + \ln g(x))\,\mathrm{d}x\right] \\ &= \exp\left[\frac{1}{y}\left(\int_0^y \ln f(x)\,\mathrm{d}x + \int_0^y \ln g(x)\,\mathrm{d}x\right)\right] \\ &= \exp\left(\frac{1}{y}\int_0^y \ln f(x)\,\mathrm{d}x\right) \cdot \exp\left(\frac{1}{y}\int_0^y \ln g(x)\,\mathrm{d}x\right) \\ &= F(y) \cdot G(y). \end{aligned}$$

3. Let $f(x) = b^x$.

$$\begin{aligned} F(y) &= \exp\left(\frac{1}{y}\int_0^y \ln f(x)\,\mathrm{d}x\right) \\ &= b^{\frac{1}{y}\int_0^y \log_b f(x)\,\mathrm{d}x} \\ &= b^{\frac{1}{y}\int_0^y \log_b b^x\,\mathrm{d}x} \\ &= b^{\frac{1}{y}\int_0^y x\,\mathrm{d}x} \\ &= b^{\frac{1}{y} \cdot \frac{y^2}{2}} \\ &= b^{\frac{y}{2}} \\ &= \sqrt{b^y}. \end{aligned}$$

4. Since $F(y) = \sqrt{f(y)}$, we notice that $f(y) = F(y)^2 = \exp\left(\frac{2}{y}\int_0^y \ln f(x)\,\mathrm{d}x\right)$, and therefore $\ln f(y) = \frac{2}{y}\int_0^y \ln f(x)\,\mathrm{d}x$.
   We substitute $g(y) = \ln f(y)$, and therefore

$$yg(y) = 2\int_0^y g(y)\,\mathrm{d}x.$$

   Therefore, differentiating both sides with respect to y gives us

$$yg'(y) + g(y) = 2g(y),$$

   and therefore

$$-g(y) + yg'(y) = 0.$$

   Multiplying y^{-2} on both sides gives us

$$-y^{-2}g(y) + y^{-1}g'(y) = 0,$$

   and therefore

$$\frac{\mathrm{d}}{\mathrm{d}y}\frac{g(y)}{y} = 0,$$

and therefore

$$\frac{g(y)}{y} = C \implies g(y) = Cy.$$

Therefore, we have

$$\begin{aligned} f(y) &= \exp g(y) \\ &= \exp(Cy) \\ &= b^y \end{aligned}$$

if we substitute $b = \exp(C) > 0$, and therefore $f(x) = b^y$ as desired.