2017.3 Question 3

By Vieta's Theorem, from the quartic equation in x, we have

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q,$$

and from the cubic equation in y, we have

$$(\alpha\beta + \gamma\delta) + (\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma) = -A.$$

Therefore, $A = -q$.

1. Since $(p, q, r, s) = (0, 3, -6, 10)$, the cubic equation is reduced to

   $$y^3 - 3y^2 - 10y + 84 = 0,$$

   and therefore

   $$(y - 2)(y - 7)(y + 6) = 0.$$

   Therefore, $y_1 = 7, y_2 = 2, y_3 = -6$, and $\alpha\beta + \gamma\delta = 7$.

2. We have

   $$\begin{aligned}(\alpha + \beta)(\gamma + \delta) &= \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta \\ &= (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) - (\alpha\beta + \gamma\delta) \\ &= q - 7 \\ &= 3 - 7 \\ &= -4.\end{aligned}$$

   By Vieta's Theorem, we have $\alpha\beta\gamma\delta = s = 10$. Therefore, $\alpha\beta$ and $\gamma\delta$ must be roots to the equation

   $$x^2 - 7x + 10 = 0.$$

   The two roots are $x = 2$ and $x = 5$, and therefore $\alpha\beta = 5$.

3. We have from the other root that $\gamma\delta = 2$.

   We notice that $(\alpha + \beta) + (\gamma + \delta) = -p = 0$. Therefore, from part 2, $(\alpha + \beta)$ and $(\gamma + \delta)$ are roots to the equation

   $$x^2 - 4 = 0.$$

   This gives us $\alpha + \beta = \pm 2$ and $\gamma + \delta = \mp 2$.

   Using the value of r and Vieta's Theorem, we have

   $$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r = 6.$$

   Plugging in $\alpha\beta = 5$ and $\gamma\delta = 2$, we have

   $$5(\gamma + \delta) + 2(\alpha + \beta) = 6.$$

   Therefore, it must be the case that $\alpha + \beta = -2$ and $\gamma + \delta = 2$.

   Hence, using the values of $\alpha\beta$ and $\gamma\delta$, α and β are solutions to the quadratic equation $x^2 + 2x + 5 = 0$, and γ and δ are solutions to the quadratic equation $x^2 - 2x + 2 = 0$.

   Solving this gives us $\alpha, \beta = -1 \pm 2i$ and $\gamma, \delta = 1 \pm i$. The solutions to the original quartic equation is

   $$x_{1,2} = -1 \pm 2i, x_{3,4} = 1 \pm i.$$