2017.3 Question 2

1. Let the complex number representing $R(P)$ be z'. Therefore,

$$\begin{aligned} z' - a &= \exp(i\theta)(z-a), \\ z' &= z\exp(i\theta) + a(1-\exp(i\theta)), \end{aligned}$$

as desired.

2. Let the complex number representing $SR(P)$ be z''. Therefore,

$$\begin{aligned} z'' - b &= \exp(i\varphi)(z'-b), \\ z'' &= z'\exp(i\varphi) + b(1-\exp(i\varphi)), \\ z'' &= [z\exp(i\theta) + a(1-\exp(i\theta))]\exp(i\varphi) + b(1-\exp(i\varphi)), \\ z'' &= z\exp(i(\theta+\varphi)) + a(1-\exp(i\theta))\exp(i\varphi) + b(1-\exp(i\varphi)). \end{aligned}$$

This will be an anti-clockwise rotation around c over an angle of $(\theta+\varphi)$, where

$$c[1-\exp(i(\theta+\varphi))] = a\exp(i\varphi) - a\exp(i(\theta+\varphi)) + b - b\exp(i\varphi),$$

If $\theta+\varphi = 2n\pi$ for some integer $n \in \mathbb{Z}$, $1-\exp(i(\theta+\varphi)) = 0$, therefore c cannot be determined. Multiplying both sides by $\exp\left(-\frac{i(\theta+\varphi)}{2}\right)$, we have

$$\begin{aligned} & c\left[\exp\left(-\frac{i(\theta+\varphi)}{2}\right) - \exp\left(\frac{i(\theta+\varphi)}{2}\right)\right] \\ &= a\left[\exp\left(\frac{i(\varphi-\theta)}{2}\right) - \exp\left(\frac{i(\theta+\varphi)}{2}\right)\right] + b\left[\exp\left(-\frac{i(\theta+\varphi)}{2}\right) - \exp\left(\frac{i(\varphi-\theta)}{2}\right)\right], \end{aligned}$$

and hence

$$\begin{aligned} -2ci\sin\left(\frac{\theta+\varphi}{2}\right) &= -2ai\exp\left(\frac{i\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) - 2bi\exp\left(-\frac{i\theta}{2}\right)\sin\left(\frac{\varphi}{2}\right), \\ c\sin\left(\frac{\theta+\varphi}{2}\right) &= a\exp\left(\frac{i\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) + b\exp\left(-\frac{i\theta}{2}\right)\sin\left(\frac{\varphi}{2}\right). \end{aligned}$$

If $\theta+\varphi = 2\pi$, we will have $z'' = z + a\exp(i\varphi) - a + b(1-\exp(i\varphi)) = z + (b-a)(1-\exp(i\varphi))$, which is a translation by $(b-a)(1-\exp(i\varphi))$.

3. If $RS = SR$, then we have

$$\begin{aligned} a(1-\exp(i\theta))\exp(i\varphi) + b(1-\exp(i\varphi)) &= b(1-\exp(i\varphi))\exp(i\theta) + a(1-\exp(i\theta)), \\ a(-1+\exp(i\varphi)+\exp(i\theta)-\exp(i(\theta+\varphi))) &= b(-1+\exp(i\varphi)+\exp(i\theta)-\exp(i(\theta+\varphi))), \\ (a-b)(1-\exp(i\varphi))(1-\exp(i\theta)) &= 0. \end{aligned}$$

Therefore, $a = b$, or $\varphi = 2n\pi$, or $\theta = 2n\pi$, for some integer $n \in \mathbb{Z}$.