2017.3 Question 13

We have

$$\begin{aligned}V(x) &= \mathrm{E}[(X-x)^2] \\ &= \mathrm{E}(X^2-2xX+x^2) \\ &= \mathrm{E}(X^2)-2x\,\mathrm{E}(X)+x^2 \\ &= \sigma^2+\mu^2-2x\mu+x^2.\end{aligned}$$

Therefore, if $Y=V(X)$, then

$$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}(V(X)) \\ &= \mathrm{E}(\sigma^2+\mu^2-2X\mu+X^2) \\ &= \sigma^2+\mu^2-2\mu\,\mathrm{E}(X)+\mathrm{E}(X^2) \\ &= \sigma^2+\mu^2-2\mu^2+\mu^2+\sigma^2 \\ &= 2\sigma^2.\end{aligned}$$

Let $X\sim U[0,1]$, we have $\mu=\mathrm{E}(X)=\frac{1}{2}$, and $\sigma^2=\mathrm{Var}(X)=\frac{1}{12}$. Therefore,

$$\begin{aligned}V(x) &= \frac{1}{12}+\frac{1}{4}-x+x^2 \\ &= x^2-x+\frac{1}{3}.\end{aligned}$$

The c.d.f. of X is F, defined as

$$\mathrm{P}(X\leq x)=F(x)=\begin{cases}0, & x\leq 0,\\ x, & 0<x\leq 1,\\ 1, & 1<x\end{cases}$$

Let the c.d.f. of Y be G, we have $G(y)=\mathrm{P}(Y\leq y)$.
Since $V([0,1])=\left[\frac{1}{12},\frac{1}{3}\right]$, we must have $G(y)=0$ for $y\leq\frac{1}{12}$ and $G(y)=1$ for $y>\frac{1}{3}$.
For $y\in\left(\frac{1}{12},\frac{1}{3}\right]$, we have

$$\begin{aligned}G(y)=\mathrm{P}(Y\leq y) &= \mathrm{P}(V(X)\leq y) \\ &= \mathrm{P}\left(\left(x-\frac{1}{2}\right)^2+\frac{1}{12}\leq y\right) \\ &= \mathrm{P}\left(\left|x-\frac{1}{2}\right|\leq\sqrt{y-\frac{1}{12}}\right) \\ &= \mathrm{P}\left(\frac{1}{2}-\sqrt{y-\frac{1}{12}}\leq x\leq\frac{1}{2}+\sqrt{y-\frac{1}{12}}\right) \\ &= F\left(\frac{1}{2}+\sqrt{y-\frac{1}{12}}\right)-F\left(\frac{1}{2}-\sqrt{y-\frac{1}{12}}\right) \\ &= \left(\frac{1}{2}+\sqrt{y-\frac{1}{12}}\right)-\left(\frac{1}{2}-\sqrt{y-\frac{1}{12}}\right) \\ &= 2\sqrt{y-\frac{1}{12}}.\end{aligned}$$

Therefore, the p.d.f. of y, g satisfies that for $y\in\left(\frac{1}{12},\frac{1}{3}\right]$,

$$g(y)=G'(y)=\frac{1}{\sqrt{y-\frac{1}{12}}}$$

and 0 everywhere else.