2017.3 Question 12

1. First, note that

$$\begin{aligned}1 &= \sum_{x,y=1}^{x=n} \mathrm{P}(X=x, Y=y)\\ &= \sum_{x=1}^{n}\sum_{y=1}^{n} k(x+y)\\ &= \sum_{x=1}^{n}\sum_{y=1}^{n}(kx+ky)\\ &= \sum_{x=1}^{n}\left(n\cdot kx + k\sum_{y=1}^{n} y\right)\\ &= nk\sum_{x=1}^{n} x + nk\sum_{y=1}^{n} y\\ &= n^2(n+1)k\end{aligned}$$

   Therefore, $k = \frac{1}{n^2(n+1)}$

$$\begin{aligned}\mathrm{P}(X=x) &= \sum_{y=1}^{n} \mathrm{P}(X=x, Y=y)\\ &= \sum_{y=1}^{n} k(x+y)\\ &= nkx + k\sum_{y=1}^{n} y\\ &= nkx + \frac{kn(n+1)}{2}\\ &= \frac{x}{n(n+1)} + \frac{1}{2n}\\ &= \frac{2x+n+1}{2n(n+1)},\end{aligned}$$

   as desired.
   By symmetry, $\mathrm{P}(Y=y) = \frac{2y+n+1}{2n(n+1)}$.
   We have

$$\mathrm{P}(X=x)\cdot\mathrm{P}(Y=y) = \frac{(2x+n+1)(2y+n+1)}{4n^2(n+1)^2}.$$

   But $\mathrm{P}(X=x, Y=y) = \frac{x+y}{n^2(n+1)}$ is not equal to this. So X and Y are not independent.

2. By definition,

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y).$$

We have

$$\begin{aligned}
\mathrm{E}(X) = \mathrm{E}(Y) &= \sum_{t=1}^{n} t \cdot \mathrm{P}(X = t) \\
&= \sum_{t=1}^{n} \frac{t \cdot (2t + n + 1)}{2n(n+1)} \\
&= \frac{1}{n(n+1)} \sum_{t=1}^{n} t^2 + \frac{1}{2n} \sum_{t=1}^{n} t \\
&= \frac{n(n+1)(2n+1)}{6n(n+1)} + \frac{n(n+1)}{4n} \\
&= \frac{2n+1}{6} + \frac{n+1}{4} \\
&= \frac{4n + 2 + 3n + 3}{12} \\
&= \frac{7n+5}{12},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}(XY) &= \sum_{x,y=1}^{n} xy \cdot \mathrm{P}(X = x, Y = y) \\
&= \sum_{x=1}^{n} \sum_{y=1}^{n} \frac{xy(x+y)}{n^2(n+1)} \\
&= \frac{1}{n^2(n+1)} \sum_{x=1}^{n} \sum_{y=1}^{n} xy(x+y) \\
&= \frac{1}{n^2(n+1)} \sum_{x=1}^{n} \sum_{y=1}^{n} (x^2 y + xy^2) \\
&= \frac{1}{n^2(n+1)} \left[\sum_{x=1}^{n} x^2 \sum_{y=1}^{n} y + \sum_{x=1}^{n} x \sum_{y=1}^{n} y^2 \right] \\
&= \frac{1}{n^2(n+1)} \cdot 2 \cdot \frac{n(n+1)(2n+1)}{6} \cdot \frac{n(n+1)}{2} \\
&= \frac{(2n+1)(n+1)}{6}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Cov}(X, Y) &= \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y) \\
&= \frac{(2n+1)(n+1)}{6} - \frac{(7n+5)^2}{144} \\
&= \frac{48n^2 + 72n + 24}{144} - \frac{49n^2 + 70n + 25}{144} \\
&= \frac{-n^2 + 2n - 1}{144} \\
&= -\frac{(n-1)^2}{144} \\
&< 0,
\end{aligned}$$

as desired.