2017.3 Question 1

1. We have

$$\begin{aligned}
\text{RHS} &= \frac{r+1}{r}\left(\frac{1}{\binom{n+r-1}{r}} - \frac{1}{\binom{n+r}{r}}\right) \\
&= \frac{r+1}{r}\left(\frac{r!(n-1)!}{(n+r-1)!} - \frac{r!n!}{(n+r)!}\right) \\
&= \frac{r+1}{r}\left(\frac{r!(n-1)!(n+r)}{(n+r)!} - \frac{r!(n-1)!n}{(n+r)!}\right) \\
&= \frac{r+1}{r}\cdot\frac{r!(n-1)!(n+r) - r!(n-1)!n}{(n+r)!} \\
&= \frac{r+1}{r}\cdot\frac{r!(n-1)!r}{(n+r)!} \\
&= \frac{(r+1)!(n-1)!}{(n+r)!} \\
&= \binom{n+r}{r+1} \\
&= \text{LHS}
\end{aligned}$$

as desired.

Therefore,

$$\begin{aligned}
\sum_{n=1}^{+\infty}\frac{1}{\binom{n+r}{r+1}} &= \sum_{n=1}^{+\infty}\frac{r+1}{r}\left(\frac{1}{\binom{n+r-1}{r}} - \frac{1}{\binom{n+r}{r}}\right) \\
&= \frac{r+1}{r}\sum_{n=1}^{+\infty}\left(\frac{1}{\binom{n+r-1}{r}} - \frac{1}{\binom{n+r}{r}}\right) \\
&= \frac{r+1}{r}\left[\sum_{n=0}^{+\infty}\frac{1}{\binom{n+r}{r}} - \sum_{n=1}^{+\infty}\frac{1}{\binom{n+r}{r}}\right] \\
&= \frac{r+1}{r}\frac{1}{\binom{0+r}{r}} \\
&= \frac{r+1}{r},
\end{aligned}$$

assuming the sum converges.

When $r = 2$, we have

$$\sum_{n=1}^{+\infty}\frac{1}{\binom{n+2}{3}} = \frac{3}{2}.$$

When $n = 1$, $\frac{1}{\binom{1+2}{3}} = \frac{1}{1} = 1$.

Therefore,

$$\sum_{n=2}^{+\infty}\frac{1}{\binom{n+2}{3}} = \frac{1}{2}$$

as desired.

2. Notice that

$$\begin{aligned}
\frac{3!}{n^3} < \frac{1}{\binom{n+1}{3}} &\iff \frac{3!}{n^3} < \frac{3!}{(n+1)n(n-1)} \\
&\iff n^3 > (n+1)n(n-1) \\
&\iff n^3 > n(n^2-1) \\
&\iff n^3 > n^3 - n \\
&\iff n > 0,
\end{aligned}$$

which is true.

Also, notice that

$$\begin{aligned}
\frac{20}{\binom{n+1}{3}} - \frac{1}{\binom{n+2}{5}} < \frac{5!}{n^3} &\iff \frac{5!}{(n+1)(n)(n-1)} - \frac{5!}{(n+2)(n+1)(n)(n-1)(n-2)} < \frac{5!}{n^3} \\
&\iff \frac{(n+2)(n-2)-1}{(n+2)(n+1)(n)(n-1)(n-2)} < \frac{1}{n^3} \\
&\iff (n^2-5)n^3 < (n^2-4)(n^2-1)n \\
&\iff n^5 - 5n^3 < n^5 - 5n^3 + 4n \\
&\iff 4n > 0,
\end{aligned}$$

which is true.

Therefore, we have that

$$\begin{aligned}
\sum_{n=3}^{+\infty} \frac{3!}{n^3} &< \sum_{n=3}^{+\infty} \frac{1}{\binom{n+1}{3}} \\
&= \sum_{n=2}^{+\infty} \frac{1}{\binom{n+2}{3}} \\
&= \frac{1}{2},
\end{aligned}$$

and therefore $\sum_{n=3}^{+\infty} \frac{1}{n^3} < \frac{1}{12}$, and $\sum_{n=1}^{+\infty} \frac{1}{n^3} < 1 + \frac{1}{8} + \frac{1}{12} = \frac{29}{24} = \frac{116}{96}$.

On the other hand, we have

$$\begin{aligned}
\sum_{n=3}^{+\infty} \frac{5!}{n^3} &< \sum_{n=3}^{+\infty} \left[\frac{20}{\binom{n+1}{3}} - \frac{1}{\binom{n+2}{5}} \right] \\
&= 20 \sum_{n=2}^{+\infty} \frac{1}{\binom{n+2}{3}} - \sum_{n=1}^{+\infty} \frac{1}{\binom{n+4}{5}} \\
&= 20 \cdot \frac{1}{2} - \frac{5}{4} \\
&= 10 - \frac{5}{4} \\
&= \frac{35}{4},
\end{aligned}$$

and therefore $\sum_{n=3}^{+\infty} \frac{1}{n^3} > \frac{7}{96}$, and $\sum_{n=1}^{+\infty} \frac{1}{n^3} > 1 + \frac{1}{8} + \frac{7}{96} = \frac{115}{96}$.

Hence,

$$\frac{115}{96} < \sum_{n=1}^{+\infty} \frac{1}{n^3} < \frac{116}{96}$$

as desired.